

Finitary Čech-de Rham Cohomology

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The present paper continues (Mallios & Raptis, *International Journal of Theoretical Physics*, 2001, **40**, 1885) and studies the curved finitary spacetime sheaves of incidence algebras presented therein from a Čech cohomological perspective. In particular, we entertain the possibility of constructing a nontrivial de Rham complex on these finite dimensional algebra sheaves along the lines of the first author's axiomatic approach to differential geometry via the theory of vector and algebra sheaves (Mallios, *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry*, Vols. 1–2, Kluwer, Dordrecht, 1998a; *Mathematica Japonica (International Plaza)*, 1998b, **48**, 93). The upshot of this study is that important "classical" differential geometric constructions and results usually thought of as being intimately associated with C^∞ -smooth manifolds carry through, virtually unaltered, to the finitary-algebraic regime with the help of some quite universal, because abstract, ideas taken mainly from sheaf-cohomology as developed in Mallios (1998a,b). At the end of the paper, and due to the fact that the incidence algebras involved have been interpreted as quantum causal sets (Raptis, *International Journal of Theoretical Physics*, 2000, **39**, 1233; Mallios & Raptis, 2001), we discuss how these ideas may be used in certain aspects of current research on discrete Lorentzian quantum gravity.

KEY WORDS: quantum gravity; causal sets; abstract differential geometry; sheaf theory; sheaf cohomology.

1. GENERAL QUESTION MOTIVATING OUR QUEST

- How much from the differential geometric panoply of C^∞ -smooth manifolds can we carry through, almost intact, to a finitary (i.e. locally finite) algebraic setting?

This is the general question that motivates the present study. We will also ponder on the following question that is closely related to the one above, but we will have to postpone our detailed elaborations about it for a future work, namely

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- Are the pathologies (e.g., the so-called singularities) of the usual differential calculus on smooth manifolds “innate” to the calculus or “differential mechanism” itself, or are they due to the particular structure (commutative algebra) sheaf of the infinitely differentiable functions that *we* employ to coordinatize the points of the C^∞ -smooth manifold?

This question, which in our opinion is the deeper of the two, puts into perspective the classical diseases in the form of infinities that assail both the classical and the quantum field theories of the dynamics of spacetime (i.e. gravity) and matter (i.e. gauge theories), theories which in turn, assume up-front a smooth base spacetime continuum on which the relevant smooth fields are localized, dynamically propagate, and interact with each other. For if these pathologies ultimately turn out to be not due to the differential mechanism itself, but rather due to our own assumption of algebras of C^∞ -smooth coordinatizations (or measurements!) of the manifold’s point events, there is certainly hope that by changing focus from the structure sheaf of rings of infinitely differentiable functions on the smooth manifold to some other “more appropriate” (or *suitable to the particular physical problem in focus*) algebra sheaves, while at the same time retaining at our disposal most (in effect, *all!*) of the powerful differential geometric constructions and techniques, the aforementioned diseases may be bypassed or even incorporated into the resulting “generalized and abstract differential calculus” (Mallios, 1998a,b), something that would effectively indicate that they are not really an essential part of “the problem” after all (Mallios, in press, Mallios, in preparation)—that is, if there is still any problem left for us to confront.⁴

So, to recapitulate our attitude towards the opening two questions: we contend that the usual differential geometry of the “classical” C^∞ -manifolds could be put into an entirely “algebraic” (i.e. sheaf-theoretic) framework, thus avoid making use of any Calculus at all, at least in the classical sense of the latter term. Thus, to a great extent, differentiability may prove to be, in a deep sense, independent of smoothness, and as a result, gravity may be transcribed to a reticular-algebraic and sheaf-theoretic environment more suitable for infusing quantum ideas into it than the problematic classical geometric C^∞ -smooth spacetime continuum. As a

⁴ Certainly, there will still remain the noble challenge to actually construct a conceptually sound and “calculationally” finite quantum theory of gravity, but at least it would have become clear that the singularities of classical gravity and the weaker but still stubborn infinities of quantum field theory are due to an inappropriate assumption—that of C^∞ -smooth coordinates, not a faulty mechanism—that of the differential calculus—and as a result they should present no essential, let alone insuperable, obstacles on our by now notoriously long (mainly due to these pathologies of the C^∞ -smooth manifold) way towards the formulation of a cogent quantum gravity. For instance, the works (Mallios and Rosinger, 1999, 2001) nicely capture this spirit, namely, that one can actually carry out the usual differential geometric constructions over spaces and their coordinate structure algebra sheaves that are very singular and anomalous—especially when viewed from the perspective of the featureless C^∞ -smooth continuum.

bonus from this transcription, we may discover that in the new finitary setting the classical smooth differential pathologies are evaded, perhaps even incorporated into the more general, abstract, and of a strong algebraic character sheaf-theoretic differential geometric picture (Mallios, 1998a,b), so that they are not essentially contributing factors to the difficulty of the problem of arriving at a sound quantum theory of gravity (Mallios, in press; Mallios, in preparation). However, it may well turn out that in the particular finitary-algebraic sheaf theories of spacetime structure and dynamics favored here, the real difficulties lie elsewhere, and that they are even more severe than the ones troubling their smooth counterparts. Undoubtedly we must keep an open mind, but then again we must also keep an optimistic eye and, at this early stage of the development of the theory, at least give such alternative combinatory-algebraic sheaf-theoretic ideas a decent chance.

We must also admit that such an endeavor is by no means new. Indeed, Einstein, as early as 1 year after he presented the general theory of relativity, doubted in the light of the quantum the very geometric smooth spacetime continuum that supported his classical field theory of gravity:

... you have correctly grasped the drawback that the continuum brings. If the molecular view of matter is the correct (appropriate) one; i.e. if a part of the universe is to be represented by a finite number of points, then the continuum of the present theory contains too great a manifold of possibilities. I also believe that this "too great" is responsible for the fact that our present means of description miscarry with quantum theory. The problem seems to me how one can formulate statements about a discontinuum without calling upon a continuum space-time as an aid; the latter should be banned from theory as a supplementary construction not justified by the essence of the problem—a construction which corresponds to nothing real. But we still lack the mathematical structure unfortunately.⁵ How much have I already plagued myself in this way of the manifold! ...” (1916) (Stachel, 1991)

and just 1 year before his death he criticized the pathological nature of the geometric spacetime continuum so that, in view of the atomistic character of Physics that the quantum revolution brought forth, he prophetically anticipated “*a purely algebraic theory for the description of reality*” (Einstein, 1956), much as follows:

... An algebraic theory of physics is affected with just the inverted advantages and weaknesses, aside from the fact that no one has been able to propose a possible logical schema for such a theory. *It would be especially difficult to derive something like a spatio-temporal quasi-order from such a schema.*⁶ I cannot imagine how the axiomatic framework for such a physics would appear, and I don’t like it when one talks about it in dark apostrophes. But I hold it entirely possible that the development will lead there; for it seems that the state of any finite spatially limited system may be fully characterized by a finite set of numbers. This seems to speak against a continuum with its infinitely many degrees of freedom. The objection is not decisive only because *one doesn’t know, in the contemporary state of mathematics, in what way the demand for*

⁵ Our emphasis

⁶ Again, our emphasis in order to prepare the reader for our quantum causal elaborations in the sequel.

*freedom from singularity (in the continuum theory) limits the manifold of solutions*⁷ ... (1954).⁸ (Stachel, 1991)

and a little bit later he agnostically admitted:

... Your objections regarding the existence of singularity-free solutions which could represent the field together with the particles I find most justified. I also share this doubt. If it should finally turn out to be the case, then I doubt in general the existence of a rational and physically useful continuous field theory. But what then? Heine's classical line comes to mind: "*And a fool waits for the answer*"... (1954). (Stachel, 1991)

So, as noted earlier, here we will content ourselves with trying to answer to the first question opening this paper and in a later work we will attempt to swim in the depths of the second (Mallios and Raptis, in preparation). Below, after we give a "crash" review of the basic ingredients in the first author's Abstract Differential Geometry (ADG) theory (Mallios, 1998a,b) (Section 2), we initiate a Čech-type of cohomological treatment of the curved finitary spacetime sheaves (finsheaves) (Raptis, 2000b) of incidence Rota algebras representing quantum causal sets (qausets) introduced in Mallios and Raptis (2001) (Section 3), and then construct the relevant de Rham complex on them based on an abstract version of de Rham's theorem *à la* ADG (Mallios, 1998a,b) (Section 4). The possibility of recovering the "classical" C^∞ -smooth Čech-de Rham complex from a net of the aforementioned finsheaf-cochains above will be entertained in Section 5. Having the finitary complex in hand, we will discuss the possibility of a finitary sheaf-cohomological classification of the reticular spin-Lorentzian connection fields \mathcal{A}_m dwelling on the gauged (i.e. curved) principal spin-Lorentzian \mathcal{G}_m -finsheaves of qausets and their associated vector (state) finsheaves studied in Mallios and Raptis (2001) in much the same way that Maxwell fields on appropriate vector (line) bundles associated with $\mathcal{G} = U(1)$ -principal fiber bundles were classified, and

⁷ Again, our emphasis.

⁸ It is quite remarkable indeed that these ideas of Einstein, especially his anticipation in the second quotation above of deriving a spatio-temporal quasi-order from a discrete-algebraic theoretical schema, foreshadow a modern approach to quantum gravity pioneered by Sorkin and coworkers coined *causal set theory* (Bombelli, *et al.* 1987; Rideout and Sorokin, 2000; Sorkin, 1990a,b, 1995, 1997, in preparation.), as well as its reticular-algebraic "*quantum causal set*" outgrowth (Mallios, 2001; Mallios and Raptis, 2001; Raptis, 2000a, 2001a). In these approaches to quantum gravity, it is fundamentally posited that underlying the spacetime manifold of macroscopic experience there are (quantum) causal set substrata—partially ordered sets (and their associated incidence algebras) with their order being regarded as the discrete and quantum ancestor of the spatio-temporal quasi order encoded in the lightcones of the classical relativistic spacetime continuum, which classical causal order, in turn, "derives" from (i.e. can be thought as a coarse descendant of) the fundamental causal order of causal sets and their quantal incidence algebraic relatives. We will return to the causal set idea as well as to its algebraic and sheaf-theoretic counterparts in some detail in Sections 6 and 7. The second author wishes to thank Rafael Sorkin for discussing the relevance of the Einstein quotation above to the problem of quantum gravity—especially to the (quantum) causal set-theoretic approach to the latter problem.

subsequently “prequantized,” along Selesnick’s line bundle axiomatics for the second quantization of bosonic (photon) fields (Selesnick, 1983), in Mallios (1998a,b, 1999, 2001) (Section 6). Arguably, as we contend in the penultimate part of the paper (section 7), sections of the vector finsheaves associated with the principal spin-Lorentzian \mathcal{G}_m -finsheaves of qausets correspond to states of “bare” or free graviton-like quanta⁹ (Mallios, 1998a,b; 1999, 2001; Mallios, in press; Mallios, in preparation; Mallios and Raptis, 2001)¹⁰, so that in the present granular-algebraic context the fool in Einstein’s quotation of Heine above will appear to have found the answer that he was desperately waiting for—and, all the more remarkably, by evading altogether the C^∞ -smoothness of the classical geometric spacetime manifold. All in all, we hold that *field and particle can possibly coexist at last* by going around the differential manifold spacetime and its pestilential singularities via discrete-algebraic and sheaf-theoretic means.¹¹ Section 7 closes with a brief discussion of some possible applications of such finitary-algebraic models and their quantum causal interpretation (Bombelli, *et al.*, 1987; Mallios and Raptis, 2001; Raptis, 2000a, 2001a,b; Sorkin, 1990a,b, 1995, 1997; Sorkin, in preparation) to current research on discrete Lorentzian quantum gravity, as well as highlighting some suggestive resemblances between our finitary application of ADG and the Kock–Lawvere Synthetic Differential Geometry (SDG) (Lavendhomme, 1996). The paper concludes (Section 8) with some physico-philosophical remarks in the spirit of the two motivating opening questions above, as it were, to close the circle that they opened.

2. BRIEF REVIEW OF ABSTRACT DIFFERENTIAL GEOMETRY

The rather technical elements from Mallios’ Abstract (Axiomatic) Differential Geometry (ADG) briefly presented below are selected from (Mallios, 1998b), which is a concise *résumé* of the more complete, but also more voluminous, work (Mallios, 1998a).¹² We itemize our brief review of ADG into four parts: the basic

⁹ Coined “causons”—the quanta of causality—in Mallios and Raptis (2001).

¹⁰ By the way, we also read from Selesnick (1983) that states of second quantized free fermionic fields can be identified with sections of Grassmannian vector bundles (correspondingly, vector sheaves in Mallios (1998a,b, 1999, 2001).

¹¹ To put it differently, and in contrast to Einstein’s mildly pessimistic premonitions above, to us field theory does not appear to be inextricably tied to a geometric spacetime continuum: one can actually do field theory on relatively discrete (i.e. “singular” and “disconnected” from the C^∞ -smooth perspective) spaces. We thus seem to abide to the general philosophy that whenever one encounters a contradiction between the mathematics (model) and the physics (reality), one should always change the maths. For *Nature cannot be pathological*; it is only that our theoretical models of Her are of limited applicability and validity (Mallios and Raptis, 2001).

¹² With the physicist in mind, we are not planning to plough through (Mallios, 1998a) in any detail here. Our “heuristic” presentation of ADG from Mallios (1998b) should suffice for the “physical level of rigour” assumed to be suitable for the present “physics oriented” study.

mathematical objects involved, the main axioms adopted, the central mathematical technique used, and ADG's core philosophy.

2.1. About the Assumptions: Three Basic Sheaves, Three Basic Objects

The basic mathematical objects involved in the development of ADG are sheaves of (complex) vector spaces V (\mathbb{C} -vector space sheaves), of (complex abelian) algebras A (\mathbb{C} -algebra sheaves), and of (differential) modules E over such algebras (A -module sheaves). These sheaves are generically symbolized by \mathcal{V} , \mathbb{A} ,¹³ and \mathcal{E} , respectively.

The first basic object to be associated with the three kinds of sheaves above is, of course, the base topological space X over which the vector space objects V dwelling in (the stalks of) \mathcal{V} , the \mathbb{C} -algebras A in \mathbb{A} , and the A -modules E in \mathcal{E} , are localized. It is one of the principal assumptions of ADG that all the three basic sheaves above have as common base space an *arbitrary topological space*—although this generality and freedom of choosing the “localization space” X is slightly constrained by assuming that it should be, at least, *paracompact* and *Hausdorff*. We will return to these two “auxiliary assumptions” for X in the next two sections. For now we note that in what follows X will be usually omitted from the sheaves above (i.e. we will simply write \mathcal{V} , \mathbb{A} etc, instead of $\mathcal{V}(X)$, $\mathbb{A}(X)$ etc) for typographical economy, unless of course we wish to comment directly on the attributes of X . At this point we should also mention that in (Mallios, 1998a,b) an open covering $\mathcal{U} = \{U \subseteq X : U \text{ open in } X\}$ of X such that an A -module sheaf $\mathcal{E}(X)$ splits locally¹⁴ into a finite n -fold Whitney (or direct) sum A^n of A with itself¹⁵ as $\mathcal{E}|_U = \mathbb{A}^n|_U \simeq \mathbb{C}\mathcal{V}^n$ ¹⁶ is called “a local frame of” or “a coordinatizing open cover of”, or even “a local choice of basis (or gauge!) for \mathcal{E}' .”¹⁷ Thus, quite reasonably, the local sections of the abelian structure \mathbb{C} -algebra sheaf \mathbb{A} relative to the local frame \mathcal{U} carry the geometric denomination “(local) coordinates,” while \mathbb{A} itself is called “the coefficient” or “ c -number coordinate sheaf” (of \mathcal{E}).

The second essential object involved in ADG is the so-called *\mathbb{C} -algebraized space*, represented by the pair (X, \mathbb{A}) ; where X is a topological space and \mathbb{A} a commutative \mathbb{C} -algebra sheaf on it. For completeness, perhaps we should also include

¹³ In both Mallios (1998a and 1998b) commutative algebra sheaves were denoted by \mathcal{A} . However, the same symbol we have already reserved for the spin-Lorentzian connections involved in Mallios and Raptis (2001). Thus, “ \mathbb{A} ” will be used henceforth to symbolize abelian algebra sheaves.

¹⁴ That is, with respect to every U in \mathcal{U} .

¹⁵ One may simply think of A^n as a finite dimensional module over A —a module of finite “rank” n .

¹⁶ Where $\mathbb{A}^n|_U \simeq \mathbb{C}\mathcal{V}^n$ denotes the corresponding n -dimensional \mathbb{C} -vector sheaf isomorphism. We also note in this context that a 1-dimensional vector sheaf (i.e. a vector sheaf of rank $n = 1$) is called a “line sheaf” in ADG.

¹⁷ Accordingly, every covering set U in \mathcal{U} is coined “a local gauge of \mathcal{E}' ”.

\mathbb{C} —the constant sheaf of complex numbers \mathbb{C} over X —into the \mathbb{C} -algebraized space, but again for typographical economy we will omit it in our elaborations below.

The last basic object involved in ADG is the so-called *differential triad*, represented by the triplet $(X, \mathcal{E}(\mathbb{A}), d)$; where X is again the base topological space, \mathbb{A} again an abelian \mathbb{C} -algebra sheaf on it,¹⁸ \mathcal{E} an A -module sheaf on it with E usually taken to be the \mathbb{Z}_+ -graded A -module $\Omega = \bigoplus_i \Omega^i$ ¹⁹ of (complex) differential forms, and d is a Cartan–Kähler type of differential operator effecting \mathcal{E} -subsheaf morphisms of the following sort: $d: \Omega^i \rightarrow \Omega^{i+1}$ (Mallios and Raptis, 2001).²⁰

2.2. About the Axioms

Essentially, the ADG theory is based on the following two axioms or assumptions:

- (a) The (abstract) de Rham complex

$$\begin{array}{ccccccc}
 0(\equiv \Omega^{-2}) & \xrightarrow{i \equiv d^{-2}} & \mathbb{C}(\equiv \Omega^{-1}) & \xrightarrow{\epsilon \equiv d^{-1}} & \mathbb{A}(\equiv \Omega^0) & & \\
 & & \xrightarrow{d^0 \equiv \partial} & \Omega^1 & \xrightarrow{d^1 \equiv d} & \Omega^2 & \xrightarrow{d^{-2}} \dots \Omega^n \xrightarrow{d^n} \dots
 \end{array} \tag{1}$$

associated with the differential triad $(X, \mathbb{A}, \mathcal{E} \equiv \Omega)$ is *exact*. In Eq. (1), \mathbb{C} is the constant sheaf of complex numbers \mathbb{C} , \mathbb{A} is a commutative \mathbb{C} -algebra (structure) sheaf, and the Ω^i s are (sub) sheaves (of Ω) of (\mathbb{Z}_+ -graded and complex) differential \mathbb{A} -modules. As we also mentioned in the previous subsection, the d^i -arrows ($i \geq 1$) linking the sheaves in the cochain expression (1) are *sheaf morphisms* and, in particular, $d^1 \equiv d$ is a nilpotent Cartan–Kähler-like differential operator²¹.

- (b) There is a *short exact exponential sheaf sequence*.²²

In this paper we are going to deal in detail only with axiom (a) (Section 4) since, as it was discussed in Section 1, we would like to study “purely

¹⁸ Which makes the doublet (X, \mathbb{A}) a *C-algebraized space* built into the differential triad.
¹⁹ With the sheaf of Ω s denoted by boldface Ω .
²⁰ The reader is referred to expression (1), where such a differential triad is put into cohomological liturgy.
²¹ Note also that our symbolism d^{-2} for the canonical injection i of the trivial constant zero sheaf $\mathbf{0} \equiv \Omega^{-2}$ into \mathbb{C} , and d^{-1} for the canonical embedding ϵ of the complex numbers into the structure algebra sheaf $\mathbb{A} \equiv \Omega^{-1}$, is a non-standard one not to be found in either Mallios (1998a or 1998b). It was adopted for “symbolic completeness” and clarity. Finally, $d^0 \equiv \partial$ is the usual partial differential operator acting in the usual way on the abelian “coordinate \mathbb{C} -algebras” dwelling in the stalks of the structure sheaf \mathbb{A} .
²² In fact, it should also be mentioned here that (a) entails any *short exact sequence* from (1) (Mallios, 1998a,b).

cohomological” features of the finsheaves in Mallios and Raptis (2001); hence, we leave the relatively secondary assumption (b) (and the one mentioned in the footnote following it) to the reader’s curiosity which, however, can be amply satisfied from reading Mallios (1998a,b). We must also remark here that, since we wish to apply ADG to the finitary regime (Sections 4–7), the assumption (a) above is not an “axiom” proper (i.e. a primitive assumption) any more; rather, it is a proposition (about the exactness of the de Rham complex) that we must actually show that it holds true in the locally finite case. We argue for this in Subsection 4.2.

2.3. About the Technique

We read from Mallios, (1998b) that the main technique employed in the *aufbau* of ADG is *sheaf-cohomology*.²³ It is fair to say that the first author’s main mathematical motivation for building ADG was the possibility of abstracting, and concomitantly generalizing, the usual de Rham cohomology of the “classical” differential calculus on C^∞ -smooth manifolds by using sheaf-cohomological techniques on vector, algebra, and (differential) module sheaves over relatively arbitrary topological spaces with ultimate aim towards resolving, or even possibly evading altogether, the singular smooth manifold theory when viewed from a broader and more potent sheaf-theoretic perspective.²⁴ Such an endeavor, that is, to generalize the usual de Rham theory, is most welcome also from a physical point of view since, and we quote von Westenholz from von Westenholz (1981), “*the structure underlying an intrinsic approach to physics is ‘essentially’ de Rham-cohomology.*” At the same time, we may recall Wheeler’s fundamental insight that in the higgledy-piggledy realm of the quantum perhaps the sole operative “principle” is one of “*law without law,*” which, in turn, can also be translated in (co)homological terms to the by now famous motto “*the boundary of a boundary is zero*” (Wheeler, 1990), and it is well known that the latter lies at the heart of (de Rham) cohomology and, as we will see in the present work, in the latter’s sheaf-theoretic generalization by ADG.

Now, on to a few slightly more technical details: a central notion in the sheaf-cohomology used in ADG is that of an \mathbb{A} -*resolution of an abstract \mathbb{A} -module sheaf* \mathcal{E} , by which one means any cochain \mathbb{A} -complex of positive degree or grade

$$\mathcal{S} : \mathbf{0} \longrightarrow \mathcal{S}^0 \xrightarrow{d^0} \mathcal{S}^1 \xrightarrow{d^1} \mathcal{S}^2 \xrightarrow{d^2} \dots \tag{2}$$

securing that the following “ \mathcal{E} -enriched” \mathbb{A} -complex

$$\tilde{\mathcal{S}} : \mathbf{0} \longrightarrow \mathcal{E} \xrightarrow{i} \mathcal{S}^0 \xrightarrow{d^0} \mathcal{S}^1 \xrightarrow{d^1} \mathcal{S}^2 \xrightarrow{d^2} \dots \tag{3}$$

²³ The reader is also referred to Strooker (1978) for a nice and relatively “down to earth” introduction to sheaf-cohomology from a modern categorical perspective.

²⁴ See Section 1.

is exact. More particularly, if the S^i s in (3) are injective \mathbb{A} -modules, the \mathbb{A} -resolution of \mathcal{E} is called “injective”. In fact, any given \mathbb{A} -module sheaf \mathcal{E} (on an arbitrary base topological space X) admits an injective resolution *à la* (3) (Mallios, 1998a). Such injective resolutions are of great importance in defining nontrivial sheaf-cohomological generalizations (or abstractions) of the concrete de Rham complex on a C^∞ -smooth manifold²⁵ and its corresponding de Rham theorem. We return to them and their use in more detail in Section 4.

2.4. About the Philosophy

We also read from Mallios (1998b) that the philosophy underlying a sheaf-theoretic approach to differential geometry, with its intrinsic, abstract in nature, sheaf-cohomological mechanism, is of an algebraic-operationalistic character. This seems to suit the general philosophy of quantum mechanics according to which what is of physical importance, the “physically real” so to speak, is less of classical ideal of “background absolute objects” (such as “spacetime”, for instance) existing “inertly out there,” independently of (i.e. not responding to) our own dynamically perturbing operations of observing “them”, and more of these operations (or dynamical actions) themselves (Finkelstein, 1996)—operations which, in turn, can be conveniently organized into algebra sheaves (Raptis, 2001a,b; Mallios and Raptis, 2001). In a nutshell, ADG has made us realize that space (time) (especially the classical, pointed geometric C^∞ -smooth model of it) is really of secondary importance for doing differential geometry; while, in practice, of primary importance are the algebraically represented (dynamical) relations between objects living on this space of which, especially in its continuous guise, we actually do not have experience anyway (Raptis and Zapartin, 2000, 2001). In view of this undermining of the smooth spacetime continuum that we wish to propound here, it is accurate to say that the central didactic point learned from ADG is that one should in a sense turn the tables around and instead of using algebras of C^∞ -functions to coordinatize (as it were, to measure!) space(time) when, as a matter of fact, these very algebras derive from the differential manifold space itself, one should rather commence with a structure algebra sheaf \mathbb{A} suitable to one’s physical problem and *derive space (time)* and possibly its (differential) geometric features from it. Algebra (ultimately, dynamics) comes first; while, space and its (differential) geometric properties second. At the same time, this “manifold neglect” that we advocate here is even more prominent in current quantum gravity research where the classical ideal of an inert, fixed, absolute, ether-like background geometrical smooth spacetime continuum (Einstein, 1924/1991), with its endogenous

²⁵ This is obtained by identifying the structure (coordinate) algebra sheaf \mathbb{A} in (1) with the algebra sheaf ${}^{\mathbb{C}}C^\infty(X)$ of infinitely differentiable \mathbb{C} -valued functions on X (i.e. $\mathbb{A}(X) \equiv {}^{\mathbb{C}}C^\infty(X)$).

pathologies and unphysical infinities, should arguably be replaced by something of a more reticular-algebraic and dynamical character (Crane, 1995; Einstein, 1936, 1956). We strongly feel that sheaf theory, especially in the intrinsically algebraic manner used by ADG, can provide a suitable language and useful tools for developing such an entirely algebraic description of quantum reality, in particular, of quantum spacetime structure and its dynamics (i.e. quantum gravity) (Mallios and Raptis, 2001).

On a more modest note, to these authors the present paper will have fulfilled a significant part of its purpose if it introduced and managed to make plain to a wider readership of (mathematical) physicists—in particular those interested in or actually working on quantum gravity—some central concepts, constructions and results from ADG, as well as how they may prove to be useful to their research. We also believe that the application of ideas from ADG to a particular finitary-algebraic context and to its associated discrete Lorentzian quantum gravity research program as in this paper, will further enhance the familiarization of the reader with the basic notions and structures of the abstract theory developed in Mallios (1998a,b, in preparation). Our locally finite, causal, and quantal version of Lorentzian gravity (Mallios and Raptis 2001; Raptis, in preparation) may be regarded as a physical toy model of ADG.

3. RUDIMENTS OF FINITARY ČECH (CO)HOMOLOGY

In this section we present the basic elements of a finitary version of the usual Čech cohomology of a (paracompact²⁶) C^∞ -smooth manifold. The epithet “finitary” pertains to a particular procedure or “algorithms”, due to Sorkin (1991), for substituting bounded regions X^{27} of C^0 -manifolds M^{28} by partially ordered sets (posets) relative to X 's *locally finite open covers*.²⁹ We further restrict our attention to “finitary poset substitutes” of X that are *simplicial complexes* (Raptis and Zapatrin, 2000, 2001; Zapatrin, to appear). Such posets, for instance, are the ones obtained from the so-called “*nerve construction*” originally due to Čech (Eilenberg and Steenrod, 1952) and subsequently, quite independently, due to Alexandrov (1956, 1961). These finitary (i.e. locally finite) simplicial skeletonizations of C^0 -manifolds will provide the essential homological backbone on which we are going

²⁶ One may wish to recall that a topological space X is said to be paracompact if every open cover of it admits a locally finite refinement (Dugundji, 1966).

²⁷ X is said to be bounded when its closure is compact. Such a space is otherwise known in the mathematical literature as “relatively compact” (Dugundji, 1966).

²⁸ Technically speaking, M is said to be C^0 -continuous when it is a topological manifold.

²⁹ An open cover or local frame \mathcal{U} of X is said to be locally finite if every point x of X has an open neighborhood that intersects a finite number of the covering open sets U in \mathcal{U} . Ultimately, every point x of X belongs to a finite number of open sets in the covering \mathcal{U} .

to support their dual finitary Rota-algebraic Čech cohomological elaborations in the sequel. So, let us commence with Sorkin’s algorithm.

3.1. Finitary \mathcal{C}^0 -Substitutes Revisited

In this section, we briefly review Sorkin’s recipe for replacing \mathcal{C}^0 -spacetime continua by finitary T_0 -poset topological spaces³⁰ relative to locally finite open coverings. The original algorithm can be found in Sorkin (1991), and in less but sufficient detail in Mallios and Raptis (2001), Raptis (2000a,b), and Raptis and Zapatin (2000, 2001).

Let X be a bounded region in a topological manifold M . Assume that $\mathcal{U} = \{U\}$ is a locally finite open cover or “coordinatizing frame” (or even “local gauge basis”³¹) of X . For every point x of X symbolize by $\Lambda(x)$ the smallest open neighborhood covering x in the subtopology \mathcal{T} of X generated by \mathcal{U} ³²: $\Lambda(x) \mid u := \cap\{U \in \mathcal{U} : x \in U\}$. Then, define the following preorder relation “ \rightarrow ”³³ between X ’s points with respect to their Λ s

$$x \rightarrow y \Leftrightarrow \Lambda(x) \subset \Lambda(y) \tag{4}$$

and enquire under what condition the “preorder-topological spaces” defined by “ \rightarrow ”³⁴ are T_0 . We read from Sorkin (1991) that this is so when “ \rightarrow ” is actually a partial order “ \dashv ”³⁵. In order to convert the aforementioned preorder-topological space into a T_0 -poset, one simply has to factor X by the following equivalence relation defined relative to “ \rightarrow ” and \mathcal{U} as

$$x \overset{\mathcal{U}}{\sim} y \Leftrightarrow \Lambda(x) = \Lambda(y) \Leftrightarrow (x \rightarrow y) \wedge (y \rightarrow x) \tag{5}$$

so that the resulting space $P(\mathcal{U}) := X / \overset{\mathcal{U}}{\sim}$, consisting of $\overset{\mathcal{U}}{\sim}$ -equivalence classes of the points of X , is a poset T_0 -topological space. This so-called finitary substitute of

³⁰ Recall that a topological space is said to be T_0 if for every pair (x, y) of points in it either x or y possesses an open neighborhood about it that does not include the other (i.e. $\forall x, y \in X, \exists \mathcal{O}(x)$ or $\mathcal{O}(y) : y \notin \mathcal{O}(x)$ or $x \notin \mathcal{O}(y)$) (Dugundji, 1966).

³¹ See again Subsection 2.1.

³² The subtopology $\mathcal{T}(\mathcal{U})$ of X is generated by arbitrary unions of finite intersections of the open U s in \mathcal{U} (viz. in other words, “the topology on the set X generated by \mathcal{U} ” or “the open sets in \mathcal{U} constitute a sub-basis for the topology \mathcal{T} of X ,” the latter being, by the hypothesis for \mathcal{U} , weaker than the initial \mathcal{C}^0 -manifold topology on X (Sorkin, 1991).

³³ We recall that a preorder is a reflexive and transitive binary relation.

³⁴ The relatively discrete topology \mathcal{T} The “ \rightarrow ” defines is based on open sets of the form $\mathcal{O}(x) = \{y : y \rightarrow x\}$ and the preorder relation $x \rightarrow y$ between X ’s points can be literally interpreted as “the constant sequence (x) converges to y in \mathcal{T} ” (Sorkin, 1991). For instance, continuous maps on the preordertopological space are exactly the ones that preserve “ \rightarrow ” (i.e. precisely the maps that preserve the convergence of the aforesaid sequences!).

³⁵ Recall that a partial order is a preorder that is also antisymmetric (i.e. $(x \rightarrow y) \wedge (y \rightarrow x) \Rightarrow x = y$).

X (Sorkin, 1991), $P(\mathcal{U})$, we will henceforth refer to as “finitary topological poset” (fintoposet) (Raptis, 2000a).

3.2. Čech–Alexandrov Nerves: Finitary Simplicial Complexes

In this subsection we present the fintoposets obtained by Sorkin’s algorithm above from a homological perspective, that is, as simplicial complexes. This presentation is based on a well known construction due to Čech (Eilenberg and Steenrod, 1952) and Alexandrov (1956, 1961), usually coined “the nerve-skeletonization of a topological manifold relative to an open cover of it”—the particular case of interest here being the nerve of a locally finite open covering \mathcal{U} of X .³⁶ Thus, an appropriate denomination for the relevant homology theory, also in keeping with the jargon of the fintoposet-discretizations of C^0 -manifolds due to Sorkin, would be “Čech–Alexandrov finitary homology’.

The specific approach to the simplicial decompositions of topological manifolds due to Čech–Alexandrov to be presented below is taken mainly from Raptis and Zapatrin (2000, 2001). In order to be able to apply concepts of simplicial homology to posets like the the fintoposets of the previous subsection, we give a relatively nonstandard definition of simplicial complexes deriving from the Čech–Alexandrov nerve construction alluded to above that effectively views them as posets. Such a definition will also come in handy in our presentation of the dual finitary-algebraic cohomological theory in the next subsection.

Thus, we first recall that the nerve \mathcal{N} of a (finitary) open cover \mathcal{U} of the bounded region X of a C^0 -manifold M is the simplicial complex having for vertices the elements of \mathcal{U} (i.e. the covering open sets) and for simplices subsets of vertices with nonempty intersections. In particular, by a k -simplex \mathcal{K} in \mathcal{N} one understands the following set of nontrivially intersecting vertices $\{U_0, \dots, U_k\}$

$$\mathcal{K} = \{U_0, \dots, U_k\} \in \mathcal{N} \Leftrightarrow U_0 \cap U_1 \cap \dots \cap U_k \neq \emptyset \tag{6}$$

Now, the nerve \mathcal{N} of a (locally finite) open cover \mathcal{U} of X , being a simplicial complex, can also be viewed as a poset P —much like in the sense of Sorkin discussed in the previous subsection.³⁷ The points of P are the simplices of the complex \mathcal{N} , and the partial order arrows “ \rightarrow ” are drawn according to the following simplicial “face rule”

$$p \rightarrow q \Leftrightarrow p \text{ is a face of } q \tag{7}$$

³⁶The reader can also refer to Dugundji (1966) for a nice introduction to the homological nerve construction.

³⁷Hence our use of the same symbol P for the (finto) posets involved in both poset constructions. Indicatively, we just note in this respect that the basic $\Lambda(x)$ involved in Sorkin’s algorithm is nothing else but the nerve simplex of x in \mathcal{U} (i.e. the open set—the smallest, in fact—in the subtopology $T(\mathcal{U})$ of X obtained by the intersection of all the U_s in \mathcal{U} that cover x —the latter collection, in case k open subsets of X in \mathcal{U} contain x , being a k -simplex in the sense described above).

As in Zapatrin (1998, to appear) and Raptis and Zapatrin (2000, 2001), we note that in the nondegenerate cases, the posets associated with the Čech-Alexandrov simplicial nerves and those derived from Sorkin’s algorithm are the same. We have chosen the homological path of nerves because their specific algebraic structure will make it possible to build the dual algebraic theory for Sorkin’s fintoposets via their so-called incidence Rota algebras in the next subsection. In turn, the latter will enable us to catch glimpses of important for our study here finitary differential and cohomological attributes that these algebras (and the finsheaves thereof) possess (Mallios and Raptis, 2001; Raptis, 2000; Raptis and Zapatrin, 2001; Zapatrin, to appear).

3.3. The “Gelfand Dual” Algebraic Theory: Čech-Type of Cohomology on Finitary Spacetime Sheaves of Incidence Algebras

Our casting Sorkin’s fintoposets in homological terms, that is, as simplicial complexes, will prove its worth in this subsection.

First, we recall from Zapatrin (1998), Raptis and Zapatrin (2000, 2001), Raptis (2000a), and Mallios and Raptis (2001) how to pass to algebraic objects dual to those finitary simplicial complexes. Such finite dimensional algebras are called incidence algebras and, in the context of enumerative combinatorics (Stanley, 1986), they were first championed by Rota (1968).³⁸

With every fintoposet P its incidence Rota algebra $\Omega(P)$ can be associated, as follows: first represent the arrows $p \rightarrow q$ in P in the Dirac operator (i.e. ket-bra) notation as $|p\rangle\langle q|$. Then define $\Omega(P)$, as a (finite dimensional) \mathbb{C} -linear space, by $\Omega(P) := \text{span}_{\mathbb{C}}\{|p\rangle\langle q| : p \rightarrow q \in P\}$, and subsequently convert it to a non-abelian \mathbb{C} -algebra by requiring closure under the following noncommutative poset-categorical (semigroup) arrow product

$$|p\rangle\langle q| \cdot |r\rangle\langle s| = |p\rangle\langle q|r\rangle\langle s| = \langle q|r\rangle \cdot |p\rangle\langle s| = \begin{cases} |p\rangle\langle s| & \text{if } q = r \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

which closes and is associative precisely because of the transitivity of the partial order “ \rightarrow ” in P .

Using the fact that Sorkin’s fintoposets are simplicial complexes naturally characterized by a positive integer-valued grade or degree (or even homological dimension (Zapatrin, to appear)³⁹), one can easily show that the corresponding $\Omega(P)$ s are \mathbb{Z}_+ -graded linear spaces (Raptis and Zapatrin, 2000; Zapatrin, to appear). With respect to this grading then, $\Omega(P)$ splits into the following direct sum

³⁸ For a beautiful introduction to incidence algebras, especially those associated with locally finite posets that are of particular interest to us here, the reader is referred to (O’Donnell and Spiegel, 1997).

³⁹ Actually, the homological dimension of a simplicial complex equals to its degree minus one.

of vectors subspaces

$$\Omega(P) = \bigoplus_{i \in \mathbb{Z}_+} \Omega^i = \Omega^0 \oplus \Omega^1 \oplus \dots := A \oplus \mathcal{R} \tag{9}$$

with $A := \Omega^0 = \text{span}_{\mathbb{C}}\{|p\rangle\langle p|\}$ a commutative subalgebra of $\Omega(P)$ consisting of its grade zero elements,⁴⁰ and $\mathcal{R} := \bigoplus_{i \geq 1} \Omega^i$ a linear subspace of $\Omega(P)$ spanned over the \mathbb{C} by elements of grade greater than or equal to one.

The crucial fact is that the correspondence $P \rightarrow \Omega(P)$ is the object-wise part of a *contravariant functor* from the category \mathfrak{P} of fintoposets and order morphisms (i.e. “fincontinuous” or “ \rightarrow ”-monotone maps (Sorkin, 1991))⁴¹ to the category \mathfrak{R} of incidence algebras and algebra homomorphisms (Zapatrin, to appear)⁴², thus it is a categorical sort of duality. In fact, the very “*Gelfand spatialization*” procedure employed in Zapatrin (to appear), Breslav and Partionov (1999), Raptis and Zapatrin (2000), and Raptis (2000a) in order to assign a topology onto the (primitive spectra consisting of kernels of equivalence classes of irreducible representations of the) $\Omega(P)$ s in such a way that they are *locally homeomorphic* to the fintoposets P from which they were derived⁴³ (Mallios and Raptis 2001; Raptis 2001a) was essentially based on this categorical duality between fintoposets and their incidence algebras. From now on we will refer to it as “*Gelfand duality*”.

It is also precisely due to the Gelfand duality between \mathfrak{P} and \mathfrak{R} that Zapatrin was able to first develop a sound *homological* theory for fintoposets or their equivalent Čech–Alexandrov nerves in \mathfrak{P} , and then to translate it to a *cohomological* theory for their corresponding incidence algebras in \mathfrak{R} (Zapatrin, to appear). For instance, a Cartan–Kähler-type of nilpotent differential operator d —arguably *the* operator to initiate a cohomological treatment of the $\Omega(P)$ s in \mathfrak{R} —was constructed (implicitly by using the Gelfand duality) from a suitable finitary version of the homological border (boundary) δ and coborder (coboundary) δ^* operators acting on the objects of \mathfrak{P} .

Indeed, with the definition of d one can straightforwardly see that the $\Omega(P)$ s in (9) are *A-modules of \mathbb{Z}_+ -graded discrete differential forms* (Raptis and Zapatrin, 2000, 2001; Zapatrin, to appear), otherwise known as *discrete differential manifolds* (Dimakis *et al.*, 1995; Dimakis and Müller-Hoissen 1999). In particular, $\Omega(P)$ ’s abelian subalgebra consisting of scalar-like quantities, $A \equiv \Omega^0$,

⁴⁰ Again, we read from Raptis and Zapatrin (2000, 2000a), and Zapatrin (to appear) that this abelian subalgebra is symbolized by \mathcal{A} , but in this study, as also alluded to in the previous section, we reserve this symbol for the spin-Lorentzian connections, and rather use A for such algebras (and \mathbb{A} for sheaves of them).

⁴¹ Or its equivalent category consisting of simplicial complexes and simplicial maps. In Raptis (2001) \mathfrak{P} was coined “the Alexandrov–Sorkin poset category.” Here we may add Čech’s contribution to it and call it “the Čech–Alexandrov–Sorkin category.”

⁴² In Raptis (2001) \mathfrak{R} was called “the Rota–Zapatrin category.”

⁴³ The reader should keep this remark in mind for what follows.

corresponds to a reticular version of the algebra ${}^{\mathbb{C}}C^{\infty}(X)$ of \mathbb{C} -valued smooth coordinates of the classical manifold's point events, while its linear subspace \mathcal{R} over A to a discrete version of the graded A -bimodule of differential forms cotangent to every point of the classical (complex) C^{∞} -smooth manifold (Raptis and Zapatrin, 2000).⁴⁴ The action of d is to effect transitions between the linear subspaces Ω^i of $\Omega(P)$ in (9), as follows: $d: \Omega^i \rightarrow \Omega^{i+1}$ (Breslav, 2000; Dimakis *et al.*, 1995; Dimakis and Müller-Hoissen, 1999; Mallios and Raptis, 2001; Raptis and Zapatrin, 2000; Zapatrin, to appear). All in all, the bonus from studying the finite dimensional incidence algebraic (cohomological) objects in \mathfrak{A} which are Gelfand dual to the fintoposet/simplicial complex (homological) objects in \mathfrak{B} is that the former encode information, in an inherently discrete guise, not only about the continuous-topological (i.e. the C^0) structure of the classical spacetime manifold like their dual correspondents P in \mathfrak{B} do (Sorkin, 1991), but also about its differential (i.e. the C^{∞}) structure (Mallios and Raptis, 2001; Raptis and Zapatrin, 2000, 2001).

Furthermore, now that we have a sort of exterior derivative operator d in our hands, all that we need to actually commence a finitary Čech-type of sheaf-cohomological study of our reticular-algebraic structures is to organize the incidence algebras into algebra sheaves and then apply to the latter ideas, techniques and results from Mallios' ADG (Mallios, 1998a,b). To this end, we first recall briefly the notion of *finitary spacetime sheaves* (finsheaves) from Raptis (2000b) and then the finsheaves of incidence algebras from Mallios and Raptis (2001).

In Raptis (2000b), finsheaves of C^0 -observables of the continuous topology of a bounded region X of a topological spacetime manifold were defined as function spaces that are *locally homeomorphic*⁴⁵ to the base fintoposet substitutes of the locally Euclidean manifold topology of X thus, technically speaking, *sheaves* over them (Mallios, 1998a,b). Subsequently in Mallios and Raptis (2001), the stalks⁴⁶ of those C^0 -finsheaves were endowed with further algebraic structure in a way that this extra structure respects the horizontal (local) "fincontinuity" (i.e. the finitary topology) of the base fintoposets—thus, ultimately, it respects the sheaf structure itself (Mallios, 1998a,b).

More specifically, finsheaves of incidence Rota algebras over Sorkin's fintoposets were defined in Mallios and Raptis (2001). We may symbolize these by $\Omega(P)$ and, as said before, omit the finitary base topological space P from its argument unless we would like to comment on it. Incidence algebras Ω dwell

⁴⁴ See also Breslav and Zapatrin (2000).

⁴⁵ See discussion around footnote 43.

⁴⁶ The stalk of a sheaf is more-or-less analogous to the fiber space of a fiber bundle—it is the point-wise (relative to the topological base space on which the sheaf is soldered) local structure of the sheaf space. For instance, as a nontopologized set, a sheaf S over a topological space X , $S(X)$, carries the discrete topology of its stalks point-wise over X , as $S(X) = \bigoplus_{x \in X} S_x$, where S_x are its stalks and the direct sum sign may also be thought of as the disjoint union operation.

in the stalks of Ω and the (germs of continuous) sections of the latter⁴⁷ inherit the algebraic structure of the Ω s for, after all, “a sheaf (of whatever algebraic objects) is its sections” (Mallios, 1998a,b)⁴⁸. Furthermore, d lifts in Ω to effect transitions between its \mathbb{Z}_+ -graded Ω^i vector subsheaves, in the following manner: $d: \Omega^i \rightarrow \Omega^{i+1}$.

For the finsheaf-cohomological aspirations of this study we note that the triplet $\mathcal{T}_m := (P_m, \Omega_m, d)$ ⁴⁹ is a finitary version of the classical (i.e. C^∞ -smooth) differential triad $\mathcal{T}_\infty := (X, \mathcal{E} \equiv {}^C\Omega_C, d)$ ⁵⁰; where X is a (bounded region of a) paracompact Hausdorff C^∞ -smooth manifold M , ${}^C\Omega_C$ is the sheaf of \mathbb{Z}_+ -graded modules of Cartan’s⁵¹ (complex⁵²) smooth (exterior) differential forms, and d is the usual nilpotent Cartan–Kähler (exterior) differential operator effecting (sub)sheaf morphisms of the form $d: \Omega^i \rightarrow \Omega^{i+1}$ (Mallios and Raptis, 2001).⁵³

In connection with the penultimate footnote, however, we note that built into the classical differential triad \mathcal{T}_∞ is the classical C^∞ -smooth \mathbb{C} -algebraized space $(X, \mathbb{A} \equiv {}^C C^\infty(X))$ over whose \mathbb{A} -structure sheaf’s objects A (i.e. the algebras of \mathbb{C} -valued C^∞ -smooth functions on X) the Cartan forms in the differential modules Ω^i superpose.⁵⁴ In fact, we emphasize from (Mallios, 1998a,b) that the entire differential calculus on smooth manifolds (i.e. the so-called “classical differential geometry”) is based on the assumption of $\mathbb{A} \equiv {}^C C^\infty(X)$ for structure sheaf of coordinates or c -coefficients⁵⁵ of the relevant differential triad, so that ADG’s power of abstracting and generalizing the classical calculus on smooth manifolds

⁴⁷ The germs of continuous sections of a sheaf \mathcal{S} by definition take values in its stalks.

⁴⁸ This gives a pivotal role to the notion of “section of a sheaf” in Mallios’ ADG, as we will also witness in the sequel.

⁴⁹ The subscript “ m ” is the so-called “finitarity or resolution index” and its (physical) meaning can be obtained directly from (Mallios and Raptis, 2001; Raptis, 2000b; Sorkin, 1991). We will use it in Section 5.

⁵⁰ As we also said in the previous footnote, that $(X, {}^C\Omega_C, d)$ has an infinite resolution index n will be explained in Section 5.

⁵¹ Hence the subscript “ C ” to the sheaf Ω .

⁵² This more or less implies that one should use a complexified manifold M , ${}^C M$, and its (co)tangent bundle $T_C^{(*)} M$ (Manin (1988); see also Subsection 6.1), but as it was also mentioned in Mallios and Raptis (2001), here we are not going to deal with the “ \mathbb{R} versus \mathbb{C} spacetime debate”.

⁵³ Interestingly enough, and in a nonsheaf-theoretic context, Zapatrin (to appear) has coined the general triple $\mathcal{D} = (\Omega, \mathcal{A}, d)$ —where Ω is a graded algebra; $\mathcal{A} \equiv \Omega^0$ an abelian subalgebra of Ω , and d a Kähler-type of differential—“a differential module \mathcal{D} over the basic algebra \mathcal{A} .” The correspondence with our (fin)sheaf-theoretic differential triads above is immediate: the latter are simply (fin)sheaves of \mathcal{D} in the sense of Zapatrin. Moreover, since d is nilpotent and we can identify in the manner of Raptis–Zapatrin (2000, 2001) and Zapatrin (to appear): $d^0 \equiv \partial: \Omega^0 \rightarrow \Omega^1$ (see 1), as well as $d^1 \equiv d^1: \Omega^1 \rightarrow \Omega^2$ and $d^2 \equiv d^2: \Omega^2 \rightarrow \Omega^3$, then the following relations are also satisfied in the finitary regime: $d^1 \circ d^0 = 0 = d^2 \circ d^1$ —a crucial condition for the exactness of the de Rham complex in (1) (Mallios, 1998a,b).

⁵⁴ The reader should refresh her memory about all these technical terms borrowed from ADG (Mallios, 1998a,b) by referring back to Subsection 3.1.

⁵⁵ See subsection 2.1.

basically lies in the possibility of assuming other more general or “exotic” (in fact, possibly more singular!) coordinates (i.e. local sections of more general abelian c -coefficient structure sheaves \mathbb{A}) while at the same time retaining *almost all* of the innate (algebraic) mechanism and techniques of classical differential geometry on smooth manifolds. All this was anticipated in Section 1.

Before we engage into some “hard core” Čech-de Rham-type of finsheaf-cohomology on the objects inhabiting the stalks of the vector, algebra, and differential module sheaves in the finitary differential triad \mathcal{T}_m in the next section, we make brief comments on the base topological spaces P_m involved in the \mathcal{T}_m s. These are Sorkin’s fintoposets and they are perfectly legitimate and admissible topological spaces on which to localize the vector, algebra, and module sheaves of our particular interest and, more importantly, to perform differential geometry *à la* ADG. For as we emphasized in Section 2, ADG is of such generality, and its concepts, constructions, and results of such a wide scope and applicability, that *in principle it admits any topological space for base space on which to solder the relevant sheaves and carry out differential geometry on them* (Mallios, 1998a,b). For example, we recall the second author’s early anticipation at the end of Raptis (2000b) (where finsheaves had just been defined!) that if one relaxed the two basic assumptions of *paracompactness* and *Hausdorffness* (or T_2 -ness⁵⁶) of ADG about the topological character of the base spaces admissible by the theory⁵⁷ to *relative compactness*⁵⁸ and T_1 -ness⁵⁹—which are precisely the two essential conditions on the C^0 -manifold X from which fintoposets P_m were derived by Sorkin’s algorithm in Sorkin (1991), then ideas of ADG could still apply to finsheaves (of whatever algebraic structures) over them. This is indeed so and, as the reader must have already noticed, it is significantly exploited in the present work.

4. FINITARY ČECH-DE RHAM SHEAF-COHOMOLOGY

This section is the nucleus of this paper. Based on an abstract version of the classical de Rham theorem on C^∞ -smooth manifolds, we entertain the possibility of a nontrivial de Rham complex on our finitary differential triad $\mathcal{T}_m = (P_m, \Omega_m, d)$. Thus, we particularize the abstract case (Mallios, 1998a,b) to our finitary regime.

⁵⁶The reader may now wish to recall that a topological space X is said to be Hausdorff, or satisfying the T_2 axiom of separation of point set topology, if for every pair of distinct points x and y in it, there exist disjoint open neighborhoods $\mathcal{O}(x)$ and $\mathcal{O}(y)$ about them (i.e. $\mathcal{O}(x) \cap \mathcal{O}(y) = \emptyset$) (Dugundji, 1966).

⁵⁷See Section 2.

⁵⁸A topological space X is called relatively compact if it is finitary in Sorkin’s sense in (Sorkin, 1991) (i.e. if it admits a locally finite open cover or frame \mathcal{U}).

⁵⁹The reader may now like to recall that a topological space X is said to be T_1 , or satisfying the first axiom of separation of point set topology, if for every pair (x, y) of points in it *both* possess open neighborhoods about them that do not include each other points (i.e. $\forall_{x,y} \in X, \exists \mathcal{O}(x), \text{ and } \mathcal{O}(y) : y \notin \mathcal{O}(x) \text{ and } x \notin \mathcal{O}(y)$) (Dugundji, 1966).

4.1. The Abstract de Rham Complex and its Theorem

In connection with the (injective) \mathbb{A} -resolution of an abstract (differential) A -module sheaf expressions (2) and (3) of Section 2, we recall from (Mallios, 1998a,b) that the n th cohomology group of an A -module sheaf $\mathcal{E}(X)$, $H^n(X, \mathcal{E})$, can be defined via its global sections $\Gamma_X(\mathcal{E}) \equiv \Gamma(X, \mathcal{E})$ as follows

$$H^n(X, \mathcal{E}) := R^n\Gamma(X, \mathcal{E}) := h^n[\Gamma(X, \mathcal{S}^\cdot)] := \ker \Gamma_X(d^n)/\text{im}\Gamma_X(d^{n-1}) \quad (10)$$

where $R^n\Gamma$ is the n th right derived functor of the global section functor $\Gamma_X(\cdot) \equiv \Gamma(X, \cdot)$.⁶⁰

Correspondingly, the abstract \mathbb{A} -complex \mathcal{S}^\cdot defined by the resolution in (2) can be directly translated by the functor Γ_X to the “global section \mathbb{A} -complex” $\Gamma_X(\mathcal{S}^\cdot)$

$$\begin{aligned} \Gamma_X(\mathcal{S}^\cdot) : \Gamma_X(\mathbf{0}) \longrightarrow \Gamma_X(\mathcal{S}^0) \xrightarrow{\Gamma_X(d^0)} \Gamma_X(\mathcal{S}^1) \xrightarrow{\Gamma_X(d^1)} \dots \\ \dots \xrightarrow{\Gamma_X(d^{n-1})} \Gamma_X(\mathcal{S}^{n-1}) \xrightarrow{\Gamma_X(d^n)} \Gamma_X(\mathcal{S}^n) \longrightarrow \dots \end{aligned} \quad (11)$$

which depicts the departure of the \mathbb{A} -differential module sheaves in it from being exact (i.e. the non-triviality of the $\mathbb{A}(X)$ -complex $\Gamma_X(\mathcal{S}^\cdot)$) (Mallios, 1998a,b).⁶¹ We coin $\Gamma_X(\mathcal{S}^\cdot)$ “the abstract de Rham complex” (ADC). We emphasize again that $\Gamma_X(\mathcal{S}^\cdot)$ is nothing more than the “section-wise analogue” of the abstract cochain complex of \mathbb{C} -vector space sheaves and \mathbb{C} -linear morphisms d^i between them that we encountered first in expression (1) and subsequently in (2) and (3).⁶²

The ADC is the main ingredient in the expression of the abstract de Rham theorem (ADT), which states, in a nutshell, that “the (sheaf) cohomology of a topological space X , with “coefficients” in some sheaf \mathcal{E} (of \mathbb{A} -modules or, more generally, of abelian groups), is that one of a certain particular \mathbb{A} -complex (canonically) associated with the given sheaf \mathcal{E} ; more precisely, the said

⁶⁰ It is rather obvious that throughout the present paper we are working in the category $\mathbf{Sh}(X)$ of sheaves (of arbitrary algebraic structures—in particular, complex differential A -modules) over X , and the functor Γ_X acts on its sheaves and the sheaf morphisms between them (in particular, on the differential sheaf morphisms d^i ; see (11) below).

⁶¹ The reader should note here that the abstract sheaf-cohomology advocated in ADG is principally concerned, via Γ_X , with the sections of the sheaves involved, thus vindicating and further exploiting the popular *motto* stated in Subsection 3.3 that “a sheaf is its sections” (see discussion around footnote 48). Thus, in connection with the philosophy of ADG (subsection 2.3), *what is of importance for ADG is more the algebraic structure of the “objects” living on “space (time)” —which algebraic structure, in turn, is conveniently captured by the corresponding algebraic relations between the sections of the respective sheaves—rather than the underlying geometric base space (time) itself.* We would like to thank the two Russian editors of Mallios (1998a), professors V. A. Lyubetsky and A. V. Zarelua, for making clear and explicit in their preface to the Russian edition of the book how ADG deals directly with the geometrical objects that live on ‘space’, thus undermining the (physical) significance of that geometric background space (time), and also how this may be of importance to current research in theoretical physics.

⁶² In Mallios (1998a,b) for instance, the ADC in (1) was coined “the abstract de Rham complex of X relative to the differential triad $(X, \Omega^1(\mathbb{A}), d \equiv \partial)$.”

cohomology is, in fact, *the cohomology of any Γ_X -acyclic resolution of \mathcal{E}^{63}* (Mallios, 1998a,b).

The abstract character of both the ADC and of the ADT that it supports consists in there being generalizations of the usual C^∞ -smooth de Rham complex and its theorem. The reader may like to recall that the classical de Rham theorem (CDT),⁶⁴ which refers to the Čech cohomology of a paracompact Hausdorff C^∞ -smooth manifold X , pertains to the cohomology of a Γ_X -acyclic resolution of the constant sheaf \mathbf{C} provided by the standard de Rham complex, which we bring forth from (1) in a slightly different form

$$\Omega_{\text{deR}}^\infty : \mathbf{0} \longrightarrow C^\infty(X) \equiv \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \longrightarrow \Omega^n \longrightarrow 0 \quad (12)$$

which complex, when \mathbf{C} -enriched,⁶⁵ provides the following exact sequence of \mathbf{C} -vector sheaves on X

$$\Omega_{\text{deR}}^\infty : \mathbf{0} \longrightarrow \mathbf{C} \longrightarrow C^\infty(X) \equiv \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \dots \longrightarrow \Omega^n \longrightarrow 0 \quad (13)$$

with n the dimensionality of the base C^∞ -manifold X .

It is well known of course that the CDT is rooted on the lemma of Poincaré which holds that *every closed C^∞ -form on X is exact*—this statement always being true at least locally (i.e. U -wise) in X . Also, we just remark here that the acyclicity of Γ_X in (12) is secured by the fact that the coordinate structure sheaf $\mathbb{A} \equiv \Omega^0 \equiv {}^{\mathbf{C}}C^\infty(X)$ is fine on X .⁶⁶

⁶³The epithet “acyclic” pertaining, of course, to the nonexactness of the ADC and the associated non-triviality of its respective cohomology groups $H^n(X, \mathcal{S}^n)$, as described above. The abstract nature of the ADT consists in that, effectively, the functor Γ_X can be substituted by *any* covariant (left exact) $\mathbb{A}(X)$ -linear functor on $\mathbf{Sh}(X)$.

⁶⁴The “C” in front of CDT could also stand for “(c)oncrete,” as opposed to the “A” (for (a)bstract) in front of ADC and its ADT.

⁶⁵Recall that \mathbf{C} is the constant sheaf of the complex numbers \mathbf{C} on X . Also, the superscript “ ∞ ” to Ω_{deR} reflects that we are dealing with the classical C^∞ -smooth case (i.e. the case of infinite finitariness or resolution index (Mallios and Raptis, 2001; Raptis, 2000b).

⁶⁶We may recall from Strooker (1978) or Mallios (1998a) that a sheaf \mathcal{S} is said to be fine if for every locally finite open covering (or every choice of coordinatizing local gauges) $\mathcal{U} = \{U_i\}$ of X there is a collection of (endo)morphisms $f_i : \mathcal{S} \rightarrow \mathcal{S}$, such that (i) $\forall_i, |f_i| := \{x \in X : (f_i)_x \neq 0\} \subset \bar{U}_i$ and (ii) $\sum_i f_i = 1$ (partition of unity). The fineness of our finsheaves is implicitly secured by their construction in Raptis (2000b) and Mallios and Raptis (2001), since, as we mentioned earlier, the region X of a C^∞ -manifold M considered there was assumed to be relatively compact (i.e. bounded) (Sorkin, 1991), as well as that it admitted locally finite open coverings \mathcal{U}_i ; hence, *in extenso*, for all practical purposes and without loss of any generality in the construction, one could assume up-front that X is, in fact, *paracompact*. The latter assumption would then immediately secure (ii) above (i.e. “partition of unity”) (Mallios, 1998a). Then, condition (i) would also be satisfied since “*every paracompact space is normal*” (Dieudonné) (Bourbaki, 1967; Dugundji, 1966), and “normality” for a topological space entails that *every locally finite open covering of it admits a “shrinking,” “precise” refinement* (Dugundji, 1966; Mallios, 1998a). Now that we have established that Ω_m^0 is fine, so are the finsheaves of graded modules of differentials over it (Mallios, 1998a). The fineness

We conclude this subsection by making the well known remark that *on a paracompact T_2 -space, sheaf-cohomology coincides with the standard Čech cohomology*, and we add that it is precisely this fact, aided by the finitary-algebraic discretizations of C^0 -manifolds à la Čech–Alexandrov–Sorkin–Zapatrin, the finsheaves thereof (Mallios and Raptis, 2001; Raptis, 2000b) and the broad sheaf-cohomological ideas of ADG, which technically conspired towards the conception of the finitary Čech-de Rham cohomology presented here. We are now in a position to present the finitary de Rham complex (FDC) and theorem (FDT).

4.2. The Finitary de Rham Complex and Theorem

We simply write the following FD $\mathbb{A} \equiv \Omega_m^0$ -complex⁶⁷ for the finitary differential triad $\mathcal{T}_m := (P_m, \Omega_m, d)$ defined in the penultimate subsection

$$\Omega_{\text{deR}}^m : \mathbf{0} \longrightarrow \Omega_m^0 \xrightarrow{d} \Omega_m^1 \xrightarrow{d} \Omega_m^2 \xrightarrow{d} \dots \longrightarrow \Omega_m^n \longrightarrow 0 \quad (14)$$

and its \mathbf{C} -enriched version

$$\Omega_{\text{deR}}^m : \mathbf{0} \longrightarrow \mathbf{C} \longrightarrow \Omega_m^0 \xrightarrow{d} \Omega_m^1 \xrightarrow{d} \Omega_m^2 \xrightarrow{d} \dots \longrightarrow \Omega_m^n \longrightarrow 0 \quad (15)$$

Both (14) and (15) depict the exactness of the finitary de Rham complex Ω_{deR}^m whose $\Gamma_{P_m}^m$ -acyclicity is secured by the fact that the Ω_m^0 -module sheaves Ω_m^1 involved in it are in fact fine by construction (Mallios and Raptis, 2001; Raptis, 2000b).⁶⁸ This is essentially the content of the FDT.⁶⁹

As it was remarked at the end of Subsection 2.2, here we will argue that the complex in (15) above (i.e. the finitary version of the abstract de Rham complex in (1)) is actually exact, thus, in effect, that the usual de Rham theory of differential forms on C^∞ -manifolds is still in force in the locally finite regime (and not merely to be taken as an axiom as (a) in Section 2.2 would *prima facie* seem to imply). Our argument is an “inverse” one as we explain below.

We consider a bounded region X of a C^∞ -smooth manifold M for which the CDT is assumed to hold. Then we employ a locally finite open gauge system \mathcal{U}_m in the sense of Sorkin (1991) to chart (coordinatize) X . Relative to \mathcal{U}_m , as we mentioned in Subsection 2.1, we extract by Sorkin’s algorithm the fintoposet P_m (Sorkin, 1991) and we build the finsheaf Ω_m (Raptis, 2000b) of incidence algebras Ω_m over it, $\Omega_m(P_m)$, as in Mallios and Raptis (2001). Then, we

of these finsheaves will play a central role in establishing the acyclicity of the corresponding Γ_X functor on the finitary de Rham complex in the next subsection.

⁶⁷ We refresh the reader’s memory by noting that the subscript “ m ” here is the finitariness or resolution index.

⁶⁸ See footnote 66.

⁶⁹ That is, that the finitary simplicial Čech cohomology of the P_m s is expressible in terms of the reticular differential forms Ω_m living on them.

know from Sorkin (1991) that the fintoposets form an inverse or projective system (Solian, 1977) poset category (net) $\mathcal{N}^{\leftarrow} := (P_m, \succeq)$, consisting of them and refinement partial order-preserving arrows \succeq between them, and which possesses an inverse or projective limit space $P_\infty = \lim P_m$ that is homeomorphic to X as $\infty \leftarrow m$. Since the Ω_m s are (categorically) $\overleftarrow{\text{dual}}$ objects to the P_m s as mentioned in Subsection 2.3 (Raptis and Zapatrin, 2000,2001; Zapatrin, 1998, to appear), they form a direct or inductive system (Solian, 1977) (again a poset category) $\mathcal{N}^{\rightarrow} := (\Omega_m, \preceq)$ consisting of the finite dimensional incidence Rota algebras Ω_m associated with the fintoposets P_m in \mathcal{N}^{\leftarrow} and injective algebra homomorphisms \preceq between them. Since the Ω_m s are discrete \mathbb{Z} -graded discrete differential manifolds, as it has been extensively argued in (Mallios and Raptis, 2001; Raptis and Zapatrin, 2000, 2001), $\mathcal{N}^{\rightarrow}$ possesses an inductive limit space, $\Omega_\infty = \lim \Omega_m$ as $m \rightarrow \infty$, that is diffeomorphic to the C^∞ -smooth region X of the C^∞ - $\overrightarrow{\text{manifold}}$ M (Raptis and Zapatrin 2000, 2001; Zapatrin, to appear). The latter effectively means that at the limit of infinite refinement of the topologies \mathcal{T}_m generated by (or having for bases) the \mathcal{U}_m s, the inductive system $\mathcal{N}^{\rightarrow}$ yields the Cartan spaces of differential forms cotangent to every point of the C^∞ -smooth X (Raptis and Zapatrin, 2000, 2001).

Now, our aforesaid “inverse” argument for the exactness of the finitary de Rham complex in (15) is based on the result that *de Rham exactness is preserved under inductive refinement*⁷⁰ since the underlying locally finite open covers \mathcal{U}_m of Sorkin may be regarded as being “good” (Bott and Tu, 1981), still by providing a *cofinal system* in the class of open coverings of X . Thus, since the exactness of the de Rham complex is assumed to hold for the inductive limit space X , it also holds for the finshaves Ω_m of reticular differential forms soldered on Sorkin’s P_m s. It must be also mentioned here that, as one would expect, Poincaré’s lemma holds locally for every contractible U in \mathcal{U}_m and, in particular, for every contractible elementary (“ur”) cell $\Lambda(x)$ covering every point x of X (see (4)),⁷¹ thus it also holds locally in every P_m of Sorkin by their very construction (Sorkin, 1991).

We close this subsection by noting that adding to the corresponding differential triads, \mathcal{T}_m and \mathcal{T}_∞ , their respective de Rham complexes, Ω_{deR}^m and Ω_{deR}^m , one obtains the following *finshavef-cohomology differential tetrads*.⁷²

- Finitary: $\mathfrak{T}_m := (P_m, \Omega_m, \mathbf{d}, \Omega_{\text{deR}}^m)$
- C^∞ -Smooth (Classical): $\mathfrak{T}_\infty := (X, \Omega, \mathbf{d}, \Omega_{\text{deR}}^m)$

These definitions and the discussion preceding them bring us to the following classical C^∞ -limit construction.

⁷⁰ Or equivalently, that *the inductive limit functor is exact* (Solian, 1977).
⁷¹ Recall that $\Lambda(x)$ is the smallest open set in the subtopology \mathcal{T}_m of X generated by the contractible open sets U in \mathcal{U}_m .
⁷² From now on this will be referred to as “fintetrad”.

5. CLASSICAL C^∞ -LIMIT CONSTRUCTION: RECOVERING THE C^∞ -SMOOTH ČECH-DE RHAM COMPLEX

The contents of the present section are effectively encoded in the following “commutative categorical limit diagram”

$$\begin{array}{ccccc}
 P_l & \xrightarrow[\text{(1)}]{\pi_l^{-1} \equiv s_l} & \mathcal{S}_l \equiv \Omega_l & \xrightarrow[\text{(2)}]{d_1} & \Omega_{\text{deR}}^l \\
 f_{lm} \downarrow (1') & & \hat{f}_{lm} \downarrow (2') & & \bar{f}_{lm} \downarrow (3') \\
 P_m & \xrightarrow[\text{(3)}]{\pi_m^{-1} \equiv s_m} & \mathcal{S}_m \equiv \Omega_m & \xrightarrow[\text{(4)}]{d_1} & \Omega_{\text{deR}}^m \\
 \vdots & & \vdots & & \vdots \\
 f_{m\infty} \downarrow (4') & & \hat{f}_{m\infty} \downarrow (5') & & \bar{f}_{m\infty} \downarrow (6')
 \end{array}$$

$$\lim_{m \rightarrow \infty} P_m \equiv P_\infty \simeq C^0(X) \xrightarrow[\text{(5)}]{\pi^{-1} \equiv s} \lim_{m \rightarrow \infty} \Omega_m \equiv \Omega \simeq^n \wedge [C^\infty(X)] \xrightarrow[\text{(6)}]{d} \Omega_{\text{deR}}^\infty$$

which we explain arrow-wise below

- (a) Arrows (1) and (3): The two horizontal arrows (1) and (3) depict the *local homeomorphism* finsheaf maps s_l and s_m , inverse to their corresponding projection maps π_l and π_m , from the fintoposet base topological spaces P_l and P_m with finitariness indices “ l ” and “ m ”, to their respective finsheaf spaces \mathcal{S}_l and \mathcal{S}_m (Raptis, 2000b). In turn, as it was shown in (Mallios and Raptis, 2001), the latter can be identified with the finsheaves Ω_l and Ω_m of the incidence Rota algebras $\Omega_l(P_l)$ and $\Omega_m(P_m)$.
- (b) Arrows (2) and (4): The two horizontal arrows (2) and (4) represent the Cartan–Kähler-like differential operators d_1 and d_m which, as said earlier, effect graded subfinsheaf morphisms, $d_1 : \Omega_l^i \rightarrow \Omega_l^{i+1}$ and $d_m : \Omega_m^i \rightarrow \Omega_m^{i+1}$, within their respective finitary de Rham finsheaf-cohomological complexes Ω_{deR}^l and Ω_{deR}^m .
- (c) Arrows (1') and (2'): The two vertical arrows (1') and (2') represent continuous injections, interpreted as topological refinements, between the fintoposets P_l and P_m ($P_l \leq P_m \Leftrightarrow f_{lm} : P_l \rightarrow P_m$) (Sorkin, 1991) and their corresponding finsheaves \mathcal{S}_l and \mathcal{S}_m ($\mathcal{S}_l \hat{\leq} \mathcal{S}_m \Leftrightarrow \hat{f}_{lm} : \mathcal{S}_l \rightarrow \mathcal{S}_m$) (Raptis, 2000b). That a continuous injection $f_{lm}(P_l \leq P_m)$ lifts to a similar continuous into map $\hat{f}_{lm}(\mathcal{S}_l \hat{\leq} \mathcal{S}_m)$ between the finsheaf spaces over the base fintoposets P_l and P_m is nicely encoded in the commutative diagram defined by the arrows [(1)–(2'), (1')–(3')] above.
- (d) Arrow (3'): The arrow \tilde{f}_{lm} represents a functor carrying (sub)sheaves and their d_1 -morphisms in the fintetrad \mathfrak{T}_l to their counterparts in the fintetrad \mathfrak{T}_m . In complete analogy with f_{lm} and \hat{f}_{lm} above, we may represent the corresponding functorial refinement relation \tilde{f}_{lm} between \mathfrak{T}_l and \mathfrak{T}_m as $\mathfrak{T}_l \tilde{\leq} \mathfrak{T}_m$.

- (e) Arrows (4') and (5'): The two arrows $f_{m\infty}$ and $\hat{f}_{m\infty}$, as the diagram symbolically depicts, are the maximal refinements obtained by subjecting the inverse or projective systems (or nets) $\mathcal{N} := (P_m, f_{lm} \equiv \leq)$ and $\hat{\mathcal{N}} := (\mathcal{S}_m \equiv \Omega_m, \hat{f}_{lm} \equiv \hat{\leq})$ to the inverse limit (categorical limit) of maximum (infinite) refinement (or localization! (Mallios and Raptis, 2001; Raptis, 2000b)) of the base fintoposets (Sorkin, 1991) and their corresponding finsheaves (Raptis, 2000b) of incidence algebras (Mallios and Raptis, 2001) (i.e. formally, as the refinement or resolution index goes to infinity “ $m \rightarrow \infty$ ”, yielding: $\lim_{m \rightarrow \infty} P_m \equiv P_\infty \simeq \mathcal{C}^0(X)$ (Raptis, 2000b; Raptis and Zapatrin, 2000; Sorkin, 1991) and $\lim_{m \rightarrow \infty} \Omega_m \equiv {}^c\Omega_\infty \equiv {}^n \bigwedge [{}^c\mathcal{C}^\infty(X)]$) (Mallios and Raptis, 2001; Raptis and Zapatrin, 2000).⁷³
- (f) Arrow (6'): The arrow (6') expresses the “convergence,” at the limit of infinite resolution, of a net $\hat{\mathcal{N}} := (\Omega_{\text{deR}}^m, \hat{\leq})$ of finitary de Rham sheaf-cochain complexes and their injective functors f_{lm} to the classical \mathcal{C}^∞ -smooth de Rham complex $\Omega_{\text{deR}}^\infty$.⁷⁴ This inverse limit convergence is in complete analogy to the projective limit convergences recalled in (f) above from (Mallios and Raptis, 2001; Raptis 2000b; Raptis and Zapatrin, 2000; Sorkin, 1991).
- (g) Arrows (5) and (6): Arrow (5) can be thought of as some kind of “injection” or “embedding” of a \mathcal{C}^0 -manifold into a \mathcal{C}^∞ -one (i.e. the well known fact in the usual Calculus that *differentiability implies continuity*, or equivalently, that every \mathcal{C}^∞ -differential manifold is a *fortiori* a \mathcal{C}^0 -topological one⁷⁵), while the arrow (6) depicts the inclusion of the sheaves Ω^i of smooth complex \mathbb{Z}_+ -graded differential forms into the smooth de Rham complex $\Omega_{\text{deR}}^\infty$ in its corresponding classical sheafcohomological differential tetrad \mathfrak{T}_∞ .

After having recovered the usual classical differential geometric \mathcal{C}^∞ -smooth structures from our reticular-algebraic substrata, we intend to initiate, at least, a sheafcohomological classification à la ADG of the nontrivial (i.e. nonflat) finitary

⁷³The reader should note above that \mathcal{N} and $\hat{\mathcal{N}}$ are the projective and inductive systems $\xrightarrow{\mathcal{N}}$ and $\xrightarrow{\hat{\mathcal{N}}}$ mentioned in the previous section, respectively. Only for notational convenience we used the same limit symbols “ $\lim_{m \rightarrow \infty}$ ” (and the same refinement relations \leq) for both the inverse and the direct limit convergence processes in $\mathcal{N} \equiv \xrightarrow{\mathcal{N}}$ and $\hat{\mathcal{N}} \equiv \xrightarrow{\hat{\mathcal{N}}}$, respectively.

⁷⁴Equivalently, and in view of (d) above, one may think of the projective system $\hat{\mathcal{N}}$ as consisting of fintetrads $\mathfrak{T}_m \hat{\leq}$ -nested by the functorial injections \hat{f}_{lm} and converging at infinite refinement to the \mathcal{C} -smooth sheaf-cohomological differential tetrad \mathfrak{T}_∞ .

⁷⁵In our finitary context, all this is just to say that the incidence algebras associated with Sorkin’s fintoposets, as well as their finsheaves, encode discrete information not only about the topological structure of “spacetime”, but also about its differentiable properties (Breslav and Zapatrin, 2000; Mallios and Raptis, 2001; Raptis, 2001a; Raptis and Zapatrin, 2000, 2001; Zapatrin, 1998, to appear).

spin-Lorentzian connections \mathcal{A}_m that were introduced and studied in Mallios and Raptis (2001). Such a possibility for classifying the spin-Lorentzian connection fields \mathcal{A}_m would amount to an effective transcription to an inherently finitary and quantal gravitational model (Mallios and Raptis, 2001) of the analogous means (i.e. techniques) and results (i.e. theorems) for classifying smooth (i.e. classical) Maxwell fields (Mallios, 1998a,b; Selesnick, 1983).

6. THE ABSTRACT WEIL INTEGRALITY AND CHERN–WEIL THEOREMS: TOWARDS A SHEAF-COHOMOLOGICAL CLASSIFICATION OF FINITARY SPIN-LORENTZIAN CONNECTIONS \mathcal{A}_m

As noted above, in this section we will attempt to emulate in a finsheaf-cohomological setting what is done in the classical C^∞ -smooth theory and entertain the possibility of assigning a cohomology class to any reticular “closed n -form”—in particular, to the (curvatures of the) finitary spin-Lorentzian 1-forms \mathcal{A}_m —dwelling in the relevant finsheaves in their respective fintetrads \mathcal{T}_m . Of great importance in such an endeavor is on the one hand ADG’s achievement of formulating abstract versions of both Weil’s integrality theorem (WIT) and of the Chern–Weil theorem (CWT) of the usual differential geometry on smooth manifolds (Mallios, 1998a,b), and on the other their possible transcription to the more concrete finitary-algebraic regime of particular interest here, for it is well known that both theorems lie at the heart of the theory of characteristic classes of classical C^∞ -smooth vector bundles and sheaves. We only translate them to our finitary-algebraic setting and, we emphasize once more, it is precisely the abstract and quite universal character of ADG that allows us to do this. However, before we present the aforesaid two theorems and their finitary versions, let us briefly inform the reader about how ADG defines and deals with “generalized differentials.” that is to say, *connections*, as well as how the latter were applied to the reticular finsheaf models in (Mallios and Raptis, 2001).

6.1. A Brief Reminder of Nontrivial Finitary \mathbb{A} -connections \mathcal{A}_m

Let us recall from (Mallios, 1998a,b) some basic sheaf-theoretic facts about abstract and general \mathbb{A} -connections before we delve into the the particular finitary case of interest here.

Let $(\mathbb{A}, \Omega, \partial)$ be a differential triad consisting, as usual, of the (commutative) \mathbb{C} -algebra structure sheaf \mathbb{A} (of “generalized coordinates”), the sheaf Ω of complex A -modules Ω (of differential forms), and the \mathbb{C} -derivation operator ∂ , which is defined as a *sheaf morphism*

$$\partial : \mathbb{A} \longrightarrow \Omega \tag{16}$$

which is also

- (i) C-linear between \mathbb{A} and Ω viewed as \mathbb{C} -vector sheaves, and
- (ii) it satisfies Leibniz’s product rule

$$\partial(s \cdot t) = s \cdot \partial t + \partial s \cdot t \tag{17}$$

which, in view of (i), implies that for every α in the constant sheaf \mathbb{C} : $\partial\alpha = 0$, or equivalently written as $\partial \mid_{\mathbb{C}} = 0$.⁷⁶ We also note that ∂ is no other than the arrow d^0 in (1) which extends to the higher grade $d^i (i \geq 1)$ sheaf morphisms in (1) when the Ω s in the sheaf Ω are \mathbb{Z}_+ -graded differential modules (defining thus graded subsheaves Ω^i of Ω).

What it must be emphasized at this point, because it lies at the heart of ADG’s sheaf-theoretic approach to differential geometry, is Kähler’s fundamental insight that

*every abelian unital ring admits a derivation map as in (16) (Mallios, 1998a,b), hence it qualifies ADG as a purely algebraic picture of differential calculus—one without any essential dependence on a “background geometrical space(time)”.*⁷⁷

For the particular finitary application of ADG here, and as it was also strongly stressed in (Mallios and Raptis, 2001), the finsheaves of incidence algebras—which effectively are ringlike structures (O’ Donnell and Spiegel, 1997)—naturally admit generalized derivations (viz. connections; see below) in the spirit of Kähler quite independently of the character of the geometric base space on which these rings are localized. This algebraic conception of derivation or connection is more in line with Leibniz’s relational intuition of this structure, rather than with Newton’s more spatial or geometrical one.⁷⁸

Now, in ADG the abstraction and generalization of the \mathbb{C} -derivation ∂ above to the notion of a (nontrivial⁷⁹) \mathbb{A} -connection \mathcal{D} is accomplished in the following

⁷⁶ Of course, it is understood that the objects “s” and “t” involved in (17) are (global) sections of \mathbb{A} .

⁷⁷ For more on this see Sections 1 and 8.

⁷⁸ We can briefly qualify this as follows: one may recall that while Newton advocated a geometric conception of derivative (e.g. as measuring the slope of the tangent line to the spatial curve, which represents the graph of the function on which this derivative operator acts), Leibniz propounded a combinatory-relational (in effect, *algebraic*) notion of derivative—one that invokes no concept of (static) ambient geometric space, but one that derives from the: (possibly *dynamical*) *relations* between the objects involved in the relational-algebraic structures in focus. He thus coined his conception of differential calculus (which he ultimately perceived as a “geometric calculus”) “*ars combinatoria*”—*combinatorial art*. In the same spirit, in ADG, with its finitary applications here and in Mallios and Raptis (2001), derivations and their abstractions-generalizations (viz. connections; see below) derive from the algebraic structure of the objects (in fact, the sections) living in the relevant (fin)sheaves (in the present paper, the incidence algebras associated with the relational fintoposets) and are not the idiosyncracies of any kind of geometric space “out there” whatsoever (see also footnote 61).

⁷⁹ The epithet “nontrivial” here pertains, as we will mention shortly, to a connection whose curvature is nonzero—commonly known as a “nonflat connection.”

two steps:

- (a) Abstraction: The abstraction of ∂ to \mathcal{D} goes briefly as follows: first, one assumes as above a differential triad $(\mathbb{A}, \Omega, \partial)$ and an \mathbb{A} -module sheaf \mathcal{E} on some topological space X ; then one defines an \mathbb{A} -connection \mathcal{D} of \mathcal{E} , as a map (in fact, again a *sheaf morphism*)

$$\mathcal{D} : \mathcal{E} \longrightarrow \mathcal{E} \otimes_{\mathbb{A}} \Omega \cong \Omega \oplus_{\mathbb{A}} \mathcal{E} \equiv (\mathcal{E}) \tag{18}$$

which again is

- (i) a \mathbb{C} -linear morphism between the \mathbb{C} -vector sheaves involved, and
- (ii) it satisfies the Leibniz rule, which now reads

$$\mathcal{D}(\beta \cdot t) = \beta \cdot \mathcal{D}(t) + t \otimes \mathcal{D}(\beta) \tag{19}$$

for any (sections) $\beta \in \mathbb{A}(U), t \in \mathcal{E}(U)$, with $U \subset X$ open (*i.e.*, properly speaking, $\beta \in \Gamma(U, \mathbb{A})$ and $t \in \Gamma(U, \mathcal{E})$).

- (b) Generalization: As briefly alluded to in the last footnote, the generalization of ∂ to \mathcal{D} basically rests on the observation made in Mallios (1998a,b) that the former is a trivial, *flat connection*,⁸⁰ so that to generalize it means, effectively, to curve it. In Mallios and Raptis (2001), for instance, the latter was accomplished by *gauging* the relevant (fin)sheaves, gauging which in turn was formally implemented by locally augmenting ∂ with a nonzero gauge potential 1-form \mathcal{A} ,⁸¹ as follows:

$$\text{Formal gauging: } \partial \longrightarrow \mathcal{D} = \partial + \mathcal{A} \tag{20}$$

We thus arrive at how physicists normally interpret a connection \mathcal{D} as a *covariant derivative*, which is a result of the process of *gauging* or *localizing* a physical structure (and its symmetries) (Mallios and Raptis, 2001). From the general perspective of a nonflat connection \mathcal{D} , the flat case ∂ is a special case recovered exactly by setting $\mathcal{A} = 0$.⁸³ Sheaf-theoretically speaking, the process of gauging or localizing means essentially that *the sheaves involved do not admit global sections* or equivalently, and perhaps more geometrically, the particular coordinate structure algebra sheaf \mathbb{A} is

⁸⁰ In a discrete context similar to the finitary one of interest to us here and to the one studied in Mallios and Raptis (2001), Dimakis and Müller–Hoissen also observed that the nilpotent Cartan–Kähler derivation $\partial \equiv d$ is a flat kind of connection (*i.e.* one whose curvature is zero).

⁸¹ This entails that the sheaf morphism \mathcal{D} in (18) is, in effect, the usual “1-form-valued assignment”: $\mathcal{D} : \mathbb{A} \rightarrow \Omega^1 \subset \Omega$ —the familiar structure encountered in the standard vector bundle models of gauge theories. See also below.

⁸² In fact, at least locally, \mathcal{D} is uniquely determined by what physicists call “the vector potential” \mathcal{A} . That is why in Mallios and Raptis (2001) we did not distinguish (at least locally) between \mathcal{D} and its nonflat part \mathcal{A} .

⁸³ But as it was emphasized in Mallios and Raptis (2001), from ADG’s perspective (Mallios, 1998a,b), ∂ is a perfectly legitimate connection; albeit, a flat or trivial one.

localized relative to the open coordinate local gauges U in \mathcal{U} covering X (Mallios, 1998a,b). Our generalized coordinatizations or measurements of the loci of events in X , as encoded in \mathbb{A} , are localized relative to the U s in \mathcal{U} . In turn, on this fact we based a finitary version of the Principle of Equivalence of general relativity on a smooth manifold and the concomitant curving of the finsheaves of incidence algebras modelling qausets in Mallios and Raptis (2001). We will return to this subsequently and in the next section.⁸⁴

We can now make the following three remarks: first, in view of the generalization or gauging of the trivial connection ∂ in the flat differential triad $(\mathbb{A}, \Omega, \mathcal{D})$, supporting the (abstract) differential tetrad whose complex is depicted in (1) to the nonflat connection \mathcal{D} in the “gauged” or “curved triad” $(\mathbb{A}, \Omega, \mathcal{D})$, and with our present sheaf-cohomological interests in mind, we read from Mallios (1998a,b) that one can define *higher order cochainprolongations* $\mathcal{D}^i (i \geq 1)$ of $\mathcal{D} (\equiv \mathcal{D}^0)$,⁸⁵ as follows

$$\begin{aligned} \Omega^0(\mathcal{E}) &\xrightarrow{\mathcal{D} \equiv \mathcal{D}^0} \Omega^1(\mathcal{E}) \xrightarrow{\mathcal{D}^1} \Omega^2(\mathcal{E}) \xrightarrow{\mathcal{D}^2} \Omega^3(\mathcal{E}) \xrightarrow{\mathcal{D}^3} \dots \\ \dots &\xrightarrow{\mathcal{D}^{i-1}} \Omega^i(\mathcal{E}) \xrightarrow{\mathcal{D}^i} \Omega^{i+1}(\mathcal{E}) \xrightarrow{\mathcal{D}^{i+1}} \dots \end{aligned} \tag{21}$$

which, in view of the fact that \mathcal{D} is nonflat, with nonzero curvature \mathcal{F} defined as

$$\mathcal{F}(\mathcal{D}) := \mathcal{D}^1 \circ \mathcal{D} \equiv \mathcal{D}^2 \neq 0, \tag{22}$$

are non-exact.⁸⁷ Thus, the obstruction of the \mathcal{D} -cochain in (21) to comprise

⁸⁴ It is worth mentioning her that the \mathbb{A} -connection \mathcal{D} to which ∂ is abstracted and generalized by (a) and (b) above, is in complete analogy to, and we quote from Kastler (1986), “the most general notion of linear connection ∇ ” used in Connes’ popular Noncommutative Differential Geometry (NDG) theory (Connes, 1994). However, the epithet “noncommutative” in Connes’ work, and in contradistinction to the first author’s ADG, pertains to also admitting *nonabelian* structure \mathbb{C} -algebra sheaves. Also in view of the noncommutative ideas in the context of qausets expounded in (Raptis, 2001a), it would certainly be worthwhile to try to relate Connes’ NDG with ADG

⁸⁵ For instance, the first order prolongation of $\mathcal{D}^0 : \mathbb{A} \rightarrow \Omega^1$ to $\mathcal{D}^1 : \Omega^1(\mathcal{E}) \rightarrow \Omega^2(\mathcal{E})$ is defined by the relation: $\mathcal{D}(u \otimes v) := u \otimes dv - v \wedge Du; u \in \mathcal{E}(U), v \in \Omega^1(U)$, and U open in X . The rest of the \mathcal{D}^i s in (21) are obtained inductively.

⁸⁶ From this definition of \mathcal{F} and the definition of \mathcal{D} above it follows that $\mathcal{F}(\mathcal{D})$ is an \mathbb{A} -morphism of the \mathbb{A} -modules involved (i.e. a structure sheaf-preserving morphism) and, in particular, a 2-form (i.e. formally: $\mathcal{F} \in Hom_{\mathbb{A}}(\mathcal{E}, \Omega^2(\mathcal{E})) = Hom_{\mathbb{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(X)$, so that when \mathcal{E} is a vector sheaf: $\mathcal{H}om_{\mathbb{A}}(\mathcal{E}, \Omega^2(\mathcal{E}))(X) = \Omega^2(End\mathcal{E}(X))$). Moreover, we read again from Mallios (1998a,b), the curvature \mathcal{F} of an \mathbb{A} -connection \mathcal{D} can be viewed as a 0-cocycle of local $n \times n$ matrices ($n = \dim \mathcal{E}$) having for entries local sections of Ω^2 (i.e. local 2-forms on X). We will return to this remark in Section 7.

⁸⁷ We bring the reader’s attention to the fact that in (22): $\mathcal{E} = \mathbb{A} \otimes_{\mathbb{A}} \mathcal{E} \equiv \Omega^0(\mathcal{E})$ (Mallios, 1998a,b). Also, interestingly enough, this definition of curvature, and in a discrete context similar to ours, was given in Dimakis and Müller-Hoissen (1999) (see also Mallios and Raptis (2001); moreover, this very definition for the curvature of an \mathbb{A} -connection was used by Connes in his NDG (Kastler, 1986; Connes, 1994).

an exact complex is essentially encoded in the nonvanishing curvature \mathcal{F} of the connection \mathcal{D} .⁸⁸ Thus, $\mathcal{F}(\mathcal{D})$ represents not only the measure of the departure from differentiating flatly, but also the deviation from setting up an (exact) cohomology based on \mathcal{D} —altogether, a measure of the departure of \mathcal{D} from nilpotence.

Second, we note in connection with the aforementioned “section-wise” spirit in which ADG is developed in (Mallios, 1998a,b) that the usual C^∞ -smooth \mathbb{A} -connection—the connection of a C^∞ -smooth manifold X the points of which are coordinatized and individuated by the coordinate algebras in the structure sheaf $\mathbb{A} = {}^C C^\infty(X)$ —acts (as a 1-form) on the sheaf of germs of sections of the (complexified) tangent bundle of $X : T_C(X) := T(X) \oplus_{\mathbb{R}} \mathbb{C}$, the latter sections being, of course, complex vector fields (i.e. first rank contravariant tensors over, that is to say, with coordinates in, ${}^C C^\infty(X)$) (Manin, 1988; Mallios, 1998a,b; Mallios and Raptis, 2001). As a matter of fact, the coordinate structure sheaf ${}^C C^\infty(X)$ of the C^∞ -manifold X is *fine*; moreover, every ${}^C C^\infty(X)$ -module sheaf \mathcal{E} (or its differential counterpart Ω) over it is also fine, hence *acyclic*. This is reflected in the well known “existence result” that *every C^∞ -smooth C -vector bundle (equivalently, C -vector sheaf) on X admits a C^∞ -connection* (Mallios, 1998a,b).

The third remark concerns a fundamental difference between a connection \mathcal{D} on a vector sheaf and its associated curvature $\mathcal{F}(\mathcal{D})$, which difference, in turn, bears on a significantly different physical interpretation that these two objects have in our finitary theory in particular and more generally in ADG.⁸⁹ We note that, while according to the definitions given above \mathcal{F} respects or preserves the abelian algebra structure (or c -coefficient) sheaf \mathbb{A} , \mathcal{D} does not (e.g. it obeys Leibniz’s rule). Since in our scheme \mathbb{A} represents *our* (local) measurements or coordinatizations relative to a (local) coordinate gauge \mathcal{U} that *we* lay out to cover and measure the events of whatever virtual geometric base “space” X *we* suppose to be “out there” suitable or convenient enough for soldering or localizing our algebraic structures, \mathcal{F} is a geometric object with respect to these measurements or “coordinatization actions” in \mathbb{A} (viz., \mathbb{A} essentially encodes the geometry of the background space X (Mallios, 1998a,b)—a kind of “ \mathbb{A} -tensor,” while \mathcal{D} cannot qualify as such.⁹⁰ All in all, \mathcal{F} (i.e. field strength) is what we measure—a geometric object with respect to our local measurements/gauge coordinates in $\mathbb{A}(X | \mathcal{U})$ —when \mathcal{D} effectively

⁸⁸ For example, section-wise in the relevant sheaves $(\mathcal{D}^{i+1} \circ \mathcal{D}^i)(s \otimes t) = t \wedge \mathcal{F}(s)$ with $s \in \Gamma(U, \mathcal{E})$, $t \in \Gamma(U, \Omega^i)$, and U open in X .

⁸⁹ This difference of interpretation between \mathcal{D} and \mathcal{F} will come in handy subsequently when we discuss and wish to interpret the Chern–Weil theorem.

⁹⁰ This is in line with what we said earlier about \mathcal{D} , namely, that it is essentially of algebraic, not geometric, character. A similar tensor/nontensor distinction is familiar to physicists that, as we noted earlier, tend to identify connection with the gauge potential part \mathcal{A} of \mathcal{D} , since, as it is well known, \mathcal{A} transforms nontensorially (i.e. inhomogeneously) under a gauge transformation, while \mathcal{F} obeys a homogeneous, tensorial gauge transformation law.

eludes them as well as the background space X supporting them (i.e. serving as a base space for the structure sheaf \mathbb{A}).

In the same train of thought, and following the (fin)sheaf-theoretic formulation of the principle of (general) covariance in Mallios and Raptis (2001), which holds that the laws of nature are equations between appropriate sheaf morphisms (the main sheaf morphisms involved being the connection and, more importantly, its curvature, which, in turn, implies that the laws of Nature are differential equations, as commonly intuited), we infer that *the laws of physics are independent of our own measurements in \mathbb{A}* , or equivalently, that *they are \mathbb{A} -covariant*.⁹¹ Also in this line of thought, we may re-raise the second question opening this paper in another manner: Is it really right to say that the laws of physics (e.g. gravity) breakdown at singularities if the latter are diseases that assail our own coordinate algebra sheaves \mathbb{A} , especially when the very mathematical expression of these laws are independent of (or covariant with) these \mathbb{A} s? Stated in a positive way: the laws of Physics cannot conceivably depend on our contingent measurements (viz., “geometries” and “spaces,” or C-algebraized spaces (X, \mathbb{A})), which in turn means that when a dynamical law appears to be singular or anomalous relative to a particular choice of “space-geometry” (X, \mathbb{A}) , the problem does not lie with the law *per se*, but, more likely, with the C-algebraized space that we have assumed.⁹² Presumably, by changing theory, ultimately, (modes or operations of) observation and (algebras of) measurements modelling the latter⁹³ (in our scheme, by changing \mathbb{A} and the base space X supporting this geometry), the apparent singularities can be resolved, for what could it possibly mean, for instance, if one could write down Einstein’s equations (as in (Mallios, in press; Mallios, in preparation)) over ultra-singular (from the C^∞ -smooth manifold viewpoint) spaces (as in (Mallios and Rosinger, 1999, 2001) other than that the law of gravity (and the differential apparatus supporting it) does not depend on the geometry of the background space(time)? We return to this caustic point in the concluding section.

So, finally, following Mallios and Raptis (2001), we are now in a position to apply ADG’s definition of \mathbb{A} -connection and define nontrivial finitary (Lorentzian)

⁹¹ In particular, for the (fin)sheaf-theoretic expression of the law of gravity in the absence of matter (i.e. the so-called vacuum Einstein equations): $\mathcal{F}_{\text{Ricci}} = 0$ (Mallios, in press; Mallios, in preparation; Raptis, in preparation), the aforesaid \mathbb{A} -covariance of $\mathcal{F}_{\text{Ricci}}$ indicates the independence of the law of gravity from our measurements (with respect to the local gauges in \mathcal{U} that we have laid out to chart X) and, ultimately, from the geometry of the background space X as the latter is encoded in the structure sheaf \mathbb{A} (Mallios, 1998a,b; Mallios and Raptis, 2001).

⁹² For instance, the singularities that assail general relativity—the classical theory of gravity—are most likely due to the assumption of coordinate algebras of infinitely differentiable functions $\mathbb{A} \equiv C^\infty(X)$ on a C^∞ -smooth spacetime manifold X , and are not the “fault” of Einstein’s equations (and the differential mechanism supporting them) whatsoever.

⁹³ In Greek, the words “theory” (“ $\theta_{\epsilon\omega\rho\iota\alpha}$ ”), “observation” (“ $\pi\alpha\rho\alpha\tau\eta\rho\eta\sigma\iota\varsigma$ ”), and “measurement” (“ $\mu \in \tau\rho\eta\sigma\iota\varsigma$ ”) go hand in hand.

$\mathbb{A}(\equiv \Omega_{\mathbf{m}}^0)$ -connections, in complete analogy with (20), as

$$\mathcal{D}_m := \partial_m + \mathcal{A}_m \tag{23}$$

on the curved (principal) finsheaves of incidence algebras⁹⁴ in their corresponding finitary differential triad $\mathcal{T}_m := (P_m, \Omega_{\mathbf{m}}, \mathfrak{d}_m)$.⁹⁵ The associated nonzero finitary curvature is denoted by $\mathcal{F}_m(\mathcal{D}_m)$.

6.2. The Abstract WIT, CWT, and Their Finitary Analogues

To make our way towards sheaf-cohomologically classifying the spin-Lorentzian \mathcal{A}_m s, we first define a de Rham p -space *à la* (Mallios, 1998a,b). This is just a paracompact Hausdorff base space X together with an exact de Rham complex as in (1) such that the latter’s cochain sequence ends at some grade $p \in \mathbb{N}$, as follows

$$\dots \longrightarrow \Omega^p \xrightarrow{d_{p=d}} d\Omega^p \longrightarrow 0 \tag{24}$$

Then, an important lemma for the (abstract) WIT⁹⁶ states that

*given such an abstract de Rham p -space, with every closed p -form ω^{97} there is associated a p -dimensional Čech cohomology class c of X with constant complex coefficients, that is, $c(\omega) \in \check{H}^p(X, \mathbb{C})$.*⁹⁸

The WIT is a particular consequence of the general lemma above by taking $p = 2$, and it states that

*every 2-dimensional integral cohomology class arises as the characteristic class of the curvature \mathcal{F} of an \mathbb{A} -connection on a line sheaf \mathcal{L} ; while, conversely, that $\mathcal{F}(\mathcal{D}) \mid_{\mathcal{L}}$ yields a (Čech) cohomology class in $\check{H}^2(X, \mathbb{Z})$.*⁹⁹

⁹⁴ In Mallios and Raptis (2001), the structure group of these \mathcal{G} -sheaves was seen to be a finitary version of the local (orthochronous) spin-Lorentz Lie group of general relativity; hence, the epithet “Lorentzian” to the $(sl(2, \mathbb{C})_m \simeq so(1, 3) \uparrow\text{-valued}) \mathcal{A}_m$ s above.

⁹⁵ The procedure that leads to (23) was coined “symmetry localization” or “gauging” in Mallios and Raptis (2001), so perhaps one could also call the corresponding triads “gauged fintriads” $\mathcal{T}_m^g := (P_m \Omega_{\mathbf{m}}, \mathcal{D}_m = \mathfrak{d}_m + \mathcal{A}_m)$.

⁹⁶ Following Mallios (1998a), we may coin this lemma “the generalized Weil Integrality theorem” for reasons to become clear shortly. Its connection with the usual WIT was first conceived in Mallios (1992).

⁹⁷ The reader may recall that $\omega \in \Omega^p$ is said to be closed when $d\omega = 0$.

⁹⁸ Even more generally, one could replace the constant coefficient sheaf \mathbb{C} by the \mathbb{C} -vector space sheaf $\ker \partial$ to arrive to the generalized WIT also employed in ADG.

⁹⁹ The connection is clear between this expression of the WIT and the generalized lemma above; in particular, the integer coefficients arise from the canonical embedding of the constant sheaf of integers \mathbb{Z} to the constant sheaf of complexes \mathbb{C} (i.e. $\mathbb{Z} \xrightarrow{\hookrightarrow} \mathbb{C}$) which, in turn, gives rise to an analogous morphism between the respective 2-dimensional sheaf-cohomology groups: $H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{C})$.

Closely related to the general and concrete WITs above, and lying at the heart of the theory of characteristic classes, is the CWT which states that

given a de Rham q -space (with q even) and a vector sheaf \mathcal{V} of rank n on X endowed with an \mathbb{A} -connection \mathcal{D} whose curvature is \mathcal{F} , if p is an invariant polynomial in $C[\lambda_{\alpha\beta}] (1 \leq \alpha, \beta \leq n)$ of degree $q/2$, then the characteristic closed q -form ω of the de Rham q -space secured by the generalized lemma for the WIT above can be obtained by identifying $\lambda_{\alpha\beta}$ with \mathcal{F}^{100} in p (i.e. from the generalized WIT in the aforesaid lemma: $c(p(\mathcal{F}(\mathcal{D}))) \equiv c(\mathcal{F}) \in \check{H}^q(X, \mathbb{C})^{101}$ and, more importantly, all this is independently of the given \mathbb{A} -connection \mathcal{D} .¹⁰²

The translation of the abstract (vector) sheaf-theoretic versions of the WIT and CWT above to our finitary case of interest is immediate: the theorems still hold true in our reticular environment, because the incidence algebra finsheaves involved fulfill all the basic technical requirements of ADG for implementing these theorems vector and algebra sheaf-theoretically (Mallios, 1998a,b; Mallios and Raptis, 2001).

6.3. En Route to Classifying the Spin-Lorentzian \mathcal{A}_m s

So far one of the main successful applications of ADG is to sheaf-cohomologically classify Maxwell fields as connections on line sheaves (Mallios, 1998a,b; Mallios, in preparation; Selesnick, 1983). Here we briefly expose this application and by analogy we speculate on a possible finsheaf-cohomological classification of the finitary spin-Lorentzian connections \mathcal{A}_m introduced in Mallios and Raptis (2001). Again, we draw information principally from Mallios (1998b).

¹⁰⁰ Where now, $\lambda_{\alpha\beta}$ are the entries of the $n \times n$ -matrix of (sections of) 2-forms constituting $\mathcal{F}(\mathcal{D})$ as defined above (Mallios, 1998a,b).

¹⁰¹ And plainly $p(\mathcal{F}) \in \wedge^q(\Omega^1(X)) \xrightarrow{c} (\wedge^q \Omega^1)(X) = \Omega^q(X)$.

¹⁰² Which pretty much vindicates the interpretational distinction that we drew earlier between the algebraic character of \mathcal{D} and the geometric character of its associated curvature $\mathcal{F}(\mathcal{D})$, since the same “effect” that we measure (viz. the geometric object \mathcal{F} which is interpreted as the field strength) can in principle arise from two different “causes” (viz. the algebraic in character \mathbb{A} -connection \mathcal{D} which is interpreted as the (gauge) potential field). The geometry (and its supporting space(time)!) that we perceive does not uniquely determine the algebraic-dynamical substratum (foam) from which it originates (by our acts of measurement). We are thus tempted, conceptually at least, to put \mathcal{D} at the quantum (algebraic) side, while \mathcal{F} at the classical (geometrical) side of the quantum divide, so that an analogue of Bohr’s correspondence principle would be that the classical (commutative) geometric realm in which \mathcal{F} lives (together with the \mathbb{A} that it respects and the space X that the latter algebras are classically supposed to coordinatize and which essentially supports \mathcal{F}) arises from measuring (i.e. “observing”) the quantum non-commutative algebraic realm (fluctuating pool, or “quantum foam” (Mallios and Rosinger, 2001; Raptis and Zapatin, 2001) from which \mathcal{D} derives and in which it varies.

The Picard Group

For the cohomological characterization of vector sheaves, ADG employs sheaf-cohomology; in particular, it uses their so-called *coordinate 1-cocycles* to classify them. So, let us dwell for a while on such a classification scheme.

First, let us assume a \mathbb{C} -algebraized space (X, \mathbb{A}) and a vector sheaf \mathcal{E} of rank or dimensionality n . Let us also assume an open cover of X or local gauge system for $\mathcal{E}(X)$, $\mathcal{U} = \{U_i\} \mid i \in I$, with respect to which one obtains the following standard Whitney-type of $\mathbb{A} \mid_{U_i}$ -isomorphisms

$$\mathcal{E}_i \equiv \mathcal{E} \mid_{U_i} \overset{\phi_i}{\cong} \mathbb{A}^n \mid_{U_i} = (\mathbb{A} \mid_{U_i})^n \equiv \mathbb{A}_i^n, i \in I^{103} \tag{25}$$

Thus, for any pair of nontrivially intersecting local open gauges U_i and U_j in \mathcal{U} (i.e. $U_i \cap U_j \neq \emptyset$) one obtains the following “local coordinate change” $\mathbb{A} \mid_{U_{ij}}$ -isomorphism

$$\phi_{ij} \equiv \phi_i \circ \phi_j^{-1} \in \text{Aut}_{\mathbb{A} \mid_{U_{ij}}}(\mathbb{A}^n \mid_{U_{ij}}) = \mathcal{GL}(n, \mathbb{A}(U_{ij})) = \mathcal{GL}(n, \mathbb{A}(U_{ij}))^{104} \tag{26}$$

In fact, such a family of local automorphisms of the \mathbb{A} -module \mathbb{A}^n provides a 1-cocycle of \mathcal{U} with coefficients in the structure group sheaf $\mathcal{GL}(n, \mathbb{A})$ in view of the relation

$$\phi_{ik} = \phi_{ij} \circ \phi_{jk}, \tag{27}$$

with $U_{ijk} \equiv U_i \cap U_j \cap U_k (i, j, k \in I)$. So, we have

$$\phi_{ij} \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbb{A})) \tag{28}$$

which is coined the *coordinate 1-cocycle of \mathcal{E} associated with the given local coordinate gauge \mathcal{U} of $\mathcal{E}(X)$* .

So, given that the first homology group of X with coefficients in $\mathcal{GL}(n, \mathbb{A})$ is (by definition) the direct limit of the corresponding Čech first cohomology group of X as the local frame \mathcal{U} ranges over all covers of X , symbolically,

$$H^1(X, \mathcal{GL}(n, \mathbb{A})) = \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{GL}(n, \mathbb{A})),^{105} \tag{29}$$

one infers that the elements of $H^1(\mathcal{U}, \mathcal{GL}(n, \mathbb{A}))$ are equivalence classes of coordinate 1-cocycles of n -dimensional vector sheaves (denoted as $[\phi_{ij}]$).

Thus, the sheaf-cohomological classification scheme for vector sheaves of rank n reads

Any n -dimensional vector sheaf \mathcal{E} on X is uniquely determined by a coordinate 1-cocycle in $Z^1(\mathcal{U}, \mathcal{GL}(n, \mathbb{A}))$ associated with any local gauge \mathcal{U} of $\mathcal{E}(X)$. We

¹⁰³ Where, $\{\phi_i\}$ is a strictly positive partition of unity of \mathbb{A} relative to $\mathcal{U} : \{\phi_i\} \subseteq \text{End } \mathbb{A} = \mathbb{A}(X)$.

¹⁰⁴ Where “ \mathcal{GL} ” denotes the n -dimensional general \mathbb{A} -linear (structure) group sheaf over X .

¹⁰⁵ That the (locally finite) open covers of X form an upwards directed set was explained in (Raptis, 2000b).

write $\Phi_{\mathbb{A}}^n(X)$ for the equivalence (isomorphism) classes of vector sheaves of rank n (i.e. $\mathbf{H}^1(X, \mathcal{GL}(n, \mathbb{A})) = \Phi_{\mathbb{A}}^n(X)$).

In keeping with the section-wise spirit in which ADG is developed, we note that the equivalence relation between two vector sheaves” classes $[\mathcal{V}_1]$ and $[\mathcal{V}_2]$ in $\Phi_{\mathbb{A}}^n(X)$ can be represented as a similarity between the section-matrices of their corresponding 1-cocycles (say, v_{ij}^1 and v_{ij}^2) relative to a common local chart \mathcal{U} covering and coordinatizing X , as follows

$$v_{ij}^2 = c_i \circ v_{ij}^1 \circ c_j^{-1} \tag{30}$$

where $c_i \in C^0(\mathcal{U}, \mathcal{GL}(n, \mathbb{A}))$ (a 0-cochain of X relative to \mathcal{U}) and $U_i \cap U_j \neq \emptyset$, as usual.

Now, in order to make direct connection, as we wish to do here, with the classification of the bosonic connections \mathcal{A}_m (viz., the “*finitary quantum causal gauge potentials*”) on the curved finsheaves of incidence algebras representing the kinematics of dynamical quantum causality in Mallios and Raptis (2001), we follow Selesnick’s axiomatics for line bundle-classification in Selesnick (1983), only here we are obviously interested in line sheaves (i.e. vector sheaves of rank 1).

So, from Mallios (1998a,b) we read that for $n = 1$ we get the following isomorphism

$$\Phi_{\mathbb{A}}^n(X) = H^1(X, \mathbb{A}^\cdot)^{106} \tag{31}$$

something that, without going into too much detail, enables us to arrive at the so-called *Picard group of X* —an abelian group consisting of equivalence classes of line bundles on X —and defined as follows

$$\text{Pic}(X) := (\Phi_{\mathbb{A}}^1(X), \otimes_{\mathbb{A}}) \equiv \Phi_{\mathbb{A}}^1(X) \tag{32}$$

where the commutative and associative tensor product functor $\otimes_{\mathbb{A}}$ has been employed to endow $\Phi_{\mathbb{A}}^n(X)$ in expression (30), and for $n = 1$, with an abelian group structure.¹⁰⁷

With the Picard group in hand, ADG achieves a sheaf-cohomological classification of Maxwell connections \mathcal{D}_{Max} on line sheaves by making use of the so-called *Chern isomorphism*:

$$H^1(X, \mathbb{A}^\cdot) = H^2(X, \mathbf{Z}) \xrightarrow{\text{WIT}} [\mathcal{F}_{Max}] \in \text{im}(H^2(X, \mathbf{Z}) \longrightarrow H^2(X, \mathbf{C})) \tag{33}$$

¹⁰⁶ Where \mathbb{A}^\cdot denotes the sheaf of invertible elements (units) in the commutative algebra structure sheaf \mathbb{A} , which means that \mathbb{A}^\cdot is an abelian group sheaf.

¹⁰⁷ Where, it is understood that the tensor product $\mathcal{L} \otimes_{\mathbb{A}} \mathcal{L}'$ of two line sheaves is a line sheaf whose coordinate 1-cocycle is the $\otimes_{\mathbb{A}}$ -product of the 1-cocycles of the corresponding line sheaves (closure with respect to $\otimes_{\mathbb{A}}$ -operation), that the inverse of a line sheaf \mathcal{L} is its dual $\mathcal{L}^{-1} = \mathcal{L}^* = \mathcal{H}om_{\mathbb{A}}(\mathcal{L}, \mathbb{A})$ (inverse), and that the neutral element is the structure sheaf \mathbb{A} itself, since $\mathcal{L} \otimes_{\mathbb{A}} \mathcal{L}^* = \mathcal{H}om_{\mathbb{A}}(\mathcal{L}, \mathcal{L}) \equiv \text{End}\mathcal{L} = \mathbb{A}$ (neutral element).

which is essentially a consequence of (the abstract version of) CWT as it translates the problem of classifying \mathcal{D}_{Max} on line sheaves *per se* to one of *finding the equivalence classes of Maxwell fields having a given curvature 2-form \mathcal{F}_{Max}* .¹⁰⁸ Moreover, in view of ADG’s quantum interpretation of connection \mathcal{D} and its curvature $\mathcal{F}(\mathcal{D})$ in Footnote 102, we may read the sheaf-cohomological classification of Maxwell fields via the Chern isomorphism above in a quantal way: what we actually determine (viz., “measure” or “observe”) is the “classical”, “commutative” (since it respects the abelian coordinatizations in \mathbb{A}) field strength \mathcal{F} , while the algebraic or quantal “causes” (or origins) of a given (measured) \mathcal{F} remain *indeterminate*, since *a given \mathcal{F} corresponds to a whole equivalence class of connections!* This indeterminacy resembles, even if only in spirit, Heisenberg’s standard one and it accords with our insistence in Footnote 102 on placing the algebraic in nature \mathcal{D} on the quantum side of Heisenberg’s *schnitt*, while its geometric in character $\mathcal{F}(\mathcal{D})$ in the classical realm on the other side of the quantum divide. For recall (a watered down version of) Bohr’s Correspondence Principle: from the noncommutative “quantum soup” we always extract (i.e. measure) commutative numbers. In fact, all this agrees with the very interpretation of the term “spacetime foam” in Mallios and Rosinger (2001) and its finitary-algebraic in Raptis and Zapatin (2001).

So, the finsheaf-cohomological classification of the nontrivial (i.e. curved) spin-Lorentzian connections \mathcal{A}_m on the principal finsheaves of gausets defined in Mallios and Raptis (2001) follows directly from the analogous classification scheme of the \mathcal{D}_{Max} above, since, as it was repeatedly stressed throughout the present paper and partly in Mallios and Raptis (2001), these finsheaves fulfill all the requirements of ADG for performing sheaf-cohomological differential geometric constructions in spite of the C^∞ -manifold. Thus, we define a “causon field”¹⁰⁹ to be the following pair

$$(\vec{\mathcal{L}}_{Caus}^m, \vec{\mathcal{D}}_{Caus}^m) \tag{34}$$

consisting of a line finsheaf $\vec{\mathcal{L}}_{Caus}^m$ associated to the curved \mathcal{G} -finsheaves $\vec{\mathcal{S}}_m$ of gausets in (Mallios and Raptis, 2001), together with a nontrivial $\vec{\Omega}_m^0$ -connection $\vec{\mathcal{D}}_{Caus}^m$ on it.¹¹⁰

¹⁰⁸ It is important to mention at this point that by a *Maxwell field* ADG means a pair $(\mathcal{L}, \mathcal{D}_{Max})$ consisting of a line sheaf \mathcal{L} and a Maxwellian \mathbb{A} -connection \mathcal{D}_{Max} on it. \mathcal{L} is interpreted as “the carrier space of \mathcal{D}_{Max} ”—and only because of the line sheaf carrying it a connection may be regarded as a geometric entity (but certainly not transformation-wise, i.e. tensorially speaking).

¹⁰⁹ In Mallios and Raptis (2001), a “causon” was defined to be the elementary particle of the “reticular bosonic spin-Lorentzian gauge potential field \mathcal{A}_m representing local curved quantum causality”, and it was speculated that it must be intimately related to the graviton—the anticipated quantum of the gravitational field.

¹¹⁰ As explained in Mallios and Raptis (2001), the arrow sign over the relevant symbols above indicate the (quantum) causal interpretation that these structures carry. From (34) it follows that the \mathcal{A}_{Caus}^m

In connection with the above, we still remark that the aforementioned “Selesnick’s Correspondence Principle” (Mallios, 2001; Mallios, in preparation; Selesnick, 1983) is used herewith in the (sheaf) topological algebra theory setup, when usually referring to the topological (*not* Banach) algebra of smooth functions on a (compact) manifold, based on a K -theory argument, providing further, directly, the “smooth analogue” of the classical (“continuous”) Serre–Swan theorem (Mallios, 2001)—or in more detail, (Mallios, in preparation). In keeping with Selesnick’s vector bundle axiomatics (Selesnick, 1983), as well as with its vector sheaf descendants (Mallios, 1998a,b,2001; Mallios, in preparation) local sections of the $\tilde{\mathcal{L}}_{Caus}^m$ s in (34) correspond to local (pre)quantum¹¹¹ states of bare or free causers. This brings us to the next section.

7. FUTURE OUTLOOK: A COUPLE OF APPLICATIONS TO DISCRETE LORENTZIAN QUANTUM GRAVITY

In this penultimate section we discuss two possible future applications of some of the ideas that were put forward above to certain aspects of current discrete Lorentzian quantum gravity research that are of interest to us.

7.1. Geometric Prequantization of Lorentzian Gravity

Continuing the remarks that conclude the last section (and footnote), we note that according to Selesnick’s general C^∞ -vector bundle axiomatics in Selesnick (1983)

local sections of line bundles correspond to states of free bosons; while, local sections of vector bundles (such as Ω^1) correspond to states of bare fermions.

ADG’s vector sheaf analogues of these results, as explained above, are of immediate avail:

boson states are sections of line sheaves, while fermion states are sections of Grassmannian (exterior) vector sheaves

and, of course, ADG’s generality allows us to consider not only smooth vector

part of $\tilde{\mathcal{D}}_{Caus}^m$ should also carry an arrow (write: $\tilde{\mathcal{A}}_{Caus}^m$) (Mallios and Raptis, 2001). Of course, we can further remark at this point that we are aware that the photon (the quantum of \mathcal{A}_{Max}) is a spin-1 gauge boson, while the graviton, a spin-2 quantum. Here, however, we do not intend to dwell longer on the spin-particulars of the causer $\tilde{\mathcal{A}}_{Caus}^m$ other than that, quantum spin-statistically speaking, it is a boson (Mallios, 1998a,b; Selesnick, 1983).

¹¹¹The epithet “prequantum” pertains to a possible application of the general theory of “geometric prequantization” as developed in Mallios (1998a,b, 1999, 2001, in preparation) to the causer field in (34). See Subsection 7.1 next.

sheaves, but *any vector sheaf over, in principle, any base space*.¹¹² An important immediate application of the foregoing ideas, and in particular of WIT, is the result, just quoted *verbatim* from Mallios (1998a,b, 1999, 2001), that

*Every free elementary particle is prequantizable; that is to say, it entails by itself a prequantizing line sheaf.*¹¹³

and in our particular finsheaves of gausets scenario for discrete Lorentzian gravity (Mallios and Raptis, 2001; Raptis, in preparation), that

A free causion entails by itself $\vec{\mathcal{L}}_{Caus}^m$.

In a nutshell, the importance of this result is that, in line with the general philosophy of geometric quantization (Mallios, 1999, 2001; Simms, 1976; Souriau, 1977; Woodhouse, 1997), one is able to arrive at the main constructions of quantum field theory (i.e. conventionally speaking, 2nd-quantized structures) by *avoiding altogether the process of 1st-quantization*, thus, effectively, by *avoiding altogether any fundamental commitment to the classical Hamiltonian mechanics and the so-called “canonical formalism” that accompanies it*. For the case of (the quantization of) gravity in particular, such a scheme (Souriau, 1977) would appear to bypass in a single leap the whole of the canonical approach to quantum gravity with all its technical and conceptual problems. Just to mention three such problems:

- (a) The problem of the diffeomorphism group $\text{Diff}(M)$ —the gauge group of general relativity—since the canonical theory assumes a background differential (i.e. C^∞ -smooth) manifold spacetime M .
- (b) The problem of finding the “right” (Hilbert) physical state space \mathcal{H} for the graviton—with the notorious problems of time, unitarity, and probability interpretation in quantum gravity that go with it.
- (c) The problem of deciding *prima facie* (i.e. straight from the classical theory in some rather “natural” way) what are (the algebras of) the physical observables (to be represented in \mathcal{H} above) relevant to quantum gravity, since, for instance, there are quantum mechanical observables without known classical counterparts (Böhm, 1979).¹¹⁴

and it is clear from the foregoing how the application of geometric prequantization *à la* ADG to a finitary, causal, and quantal version of Lorentzian gravity (Mallios, 2001; Mallios and Raptis, 2002b) may be able to evade all three. At this point we could also infer that the finsheaf-theoretic scenario for discrete Lorentzian

¹¹² Which serves as a base space(time) (viz. “configuration space”) for the physical system in focus.

¹¹³ In fact, this is so regardless of whether the elementary particle is a boson or fermion (Mallios, 1992, 1998a,b, 1999, 2001; Selesnick, 1983).

¹¹⁴ Thus, it would be begging the question to (canonically) quantize a classical theory—in particular, general relativity—since we could encounter entities in the quantum regime that are not observable at the classical level (in which case, the correspondence principle would be effectively meaningless).

quantum gravity via ADG is more in line with a covariant path integral (over spaces of self-dual $sl(2, \mathbb{C})_m$ -valued \mathcal{A}_m s) approach to the quantization of gravity, rather than with the canonical (Hamiltonian) scheme. This too was anticipated at the end of Mallios and Raptis (2001).

7.2. Finitary ADG on Consistent-Histories, Topoi in Quantum Logic and Quantum Gravity, and a Connection with SDG

The second future application of the finitary ADG ideas above that we would like to suggest is to the consistent-histories approach to quantum theory and quantum gravity in particular.

In Raptis (submitted), for instance, sheaves of qausets over the Vietoris-topologized base poset category of Boolean subalgebras of the universal orthoalgebra of history propositions were introduced, as it were, to define *sheaves of quantum causal histories*. At the end of the paper it was speculated that one should be able to do differential geometry *à la* ADG on such sheaves—something that could be of immediate value to quantum gravity research when approached via consistent-histories. There seems to be no foreseeable obstacle to such an endeavor, since, as we have time and again stressed, the results of ADG are effectively base space independent.¹¹⁵

A more specific project along these lines could be the following: since the topos-theoretic perspective on both the quantal logic of consistent-histories (Isham, 1997) and on the usual quantum logic (Butterfield *et al.*, 2000; Butterfield and Isham, 1998,1999) has revealed to us that in a very geometrical sense quantum logic is warped or curved relative to its local classical sublogics, so the closely analogous topos-like aggregate of quantum causal histories” sheaves¹¹⁶ may also exemplify such a curvature which, in view of the quantum causal interpretation of the objects in the QCHT, may be directly related to the reticular curved quantum causality (viz., discrete Lorentzian quantum gravity) studied in Mallios and Raptis (2001).¹¹⁷ Thus, for example, it would be interesting to *search for a nontrivial characteristic cocycle in the curved sheaves of quantum causal histories*. The ideas developed in this paper clearly indicate that this is a legitimate and quite feasible project.¹¹⁸

¹¹⁵ The reader is encouraged to read the concluding remarks in Raptis (submitted) that predict, for example, a possible sheaf-cohomological classification of the algebra sheaves of quantum causal histories along the lines of ADG. See also the following paragraph.

¹¹⁶ Coined the “Quantum Causal Histories Topos” (QCHT) in Raptis (submitted).

¹¹⁷ That a topos-theoretic approach not only to quantum logic, but also to quantum gravity proper, is quite a promising route, was nicely presented in Butterfield and Isham (2000). See also Raptis (2001b).

¹¹⁸ This project, in the context of the topos-theoretic approach to quantum logic proper (Butterfield *et al.*, 2000; Butterfield and Isham, 1998,1999) and to the similar approach to the logic of

We close this section with two remarks: first, the aforementioned possible QCHT organization of the sheaves of quantum causal histories should be compared with the topoi modelling the mathematical universes in which to carry out the Kock–Lawvere Synthetic Differential Geometry (SDG) (Lavendhomme, 1996). The latter, in a nutshell, is an extension of the usual, “classical” C^∞ -differential geometry by two means: first, by admitting nilpotent “real numbers”, and second, by suitably modifying the logic underlying the usual Calculus from the Boolean (classical) one of the topos **Set** of classical constant sets (in which, for instance, the usual C^∞ -calculus is constructed), to the Brouwerian (intuitionistic) one of the topos **Sh**(X) of varying sets (Raptis, submitted) in order to cope with the first extension.¹¹⁹ Moreover, SDG purports to be able to translate almost all the basic constructions of the usual Calculus on smooth manifolds into synthetic terms.¹²⁰ Only for this, and in view of similar claims made about ADG in the present paper, it would certainly be worthwhile to initiate a comparison between ADG and SDG, even if only at an abstract mathematical level.¹²¹ However, as far as applications to quantum gravity are concerned, such a comparison could prove to be beneficial to physics too, since it has been seriously proposed that SDG could cast light on the problem of quantum gravity (Butterfield and Isham, 2000).

Finally, Finkelstein (1988), in a reticular-algebraic model for the quantum structure and dynamics of spacetime similar to ours, called the “causal net,” urged us to develop a causal version of the (co)homology theory of the usual algebraic topology—as it were, “*to algebraize and causalize (with ultimate aim to quantize) topology (in order to apply it to the quantum structure and dynamics of spacetime).*” Since, following Sorkin’s insight (Sorkin 1995) to change physical interpretation of the fintoposets involved from topological to causal, our finitary incidence algebras model qausets (and not topological spaces proper) (Raptis, 2000a) while their curved finsheaves represent the qausets’ dynamical variations (Mallios and Raptis, 2001), it is perhaps fair to say that the finitary Čech-de Rham finsheaf-cohomology presented in the present paper comes very close to materializing Finkelstein’s imperative above.¹²²

consistent-histories (Isham, 1997), was originally conceived by John Hamilton and Chris Isham (Chris Isham in private communication).

¹¹⁹ It is worth mentioning here the result, due to Dubuc (1979, 1981), that the category of (finite dimensional) paracompact C^∞ -smooth manifolds and diffeomorphisms between them can be faithfully embedded into a topos, preserving fiber products, open covers, as well as mapping the usual real line \mathbb{R} into the aforementioned nilpotent-enriched “real numbers”—the so-called Kock–Lawvere ring R_{KL} .

¹²⁰ For instance, it has provided, like we have done here for the finitary case, a synthetic version of de Rham’s theorem at the level of chain complexes, and much more . . .

¹²¹ For instance, it would be interesting to compare the way the two theories extend and generalize the usual de Rham theory on C^∞ -smooth manifolds.

¹²² Thus, the reader can now go back to the various (co)homological structures mentioned in the present paper and draw an arrow (indicating causal, not topological proper, interpretation) over their symbols!

8. PHYSICO-PHILOSOPHICAL FINALE

We close the present paper by making some physico-philosophical remarks in the spirit of the two questions raised in the introductory section.

We hope that by this work we have made it clear that one can actually carry out most of the usual differential geometric constructions effectively without use of any sort of C^∞ -smoothness or any of the conventional “classical” Calculus that goes with it. This to a great extent indicates, in partial response to the second question opening the paper, that the differential geometric technique or “mechanism”—the “differential mechanics” or “differential operationalism” so to speak—is not crucially dependent on a C^∞ -smooth background space(time) and the coordinate algebras of C^∞ -smooth functions (or generalized “position measurements”, or even “localizations”) associated with its geometric points, no matter how strongly the usual calculus on manifolds has “forced” us so far to postulate it up-front before we set up any differential geometric theory/model of Nature. To this in many ways misleading pseudo-imperativeness we tend to attribute the almost instinctive reaction of the modern mathematical physicist to regard the smooth continuum as a model of spacetime of great *physical* significance and import.¹²³ Admittedly, the manifold has served us well; after all, the very differential geometry on which Einstein’s successful general relativity theory of the (classical) gravitational field rests is vitally dependent on it.

However, it soon became clear by means of the celebrated singularity theorems (Penrose, 1977) that the classical theory of gravity and the smooth spacetime continuum that supports it are assailed by anomalies and diseases in the form of singularities long before a possible quantization scheme for them becomes an issue. Especially the so-called black hole singularities seem to indicate that general relativity and its classical continuous spacetime backbone break down near, let alone in the interior of, them.¹²⁴ It now appears plain to us that classical differential geometry cannot cope with such pathologies and this has prompted theoretical physicists to speculate that a quantum theory of gravity should be able to heal or at least alleviate these maladies. Indeed, Hawking’s semi-classical (or semi-quantum!) treatment of these objects showed us that they should properly be regarded not as universal absorbers, but as some kind of thermodynamically unstable black bodies that can thermally radiate quanta (Hawking, 1975). An even more startling behavior of such singularities was discovered a bit earlier by Bekenstein and Hawking (Bekenstein, 1973; Hawking, 1976) who showed that they have rich

¹²³ This brings to mind Einstein’s famous suspicion about the actual physicality of spacetime (“*Space and time are concepts by which we think, not conditions in which we live*” (Einstein, 1950)), and its C^∞ -smooth manifold model (see the three quotations of Einstein in the opening section).

¹²⁴ Furthermore, this came to be distilled to the following Popperian “falsifiability”-like motto: general relativity is a good theory, because, among other things such as agreement with experiments/observations, it predicts its own downfall by the existence of singular solutions to Einstein’s gravitational field equations.

thermodynamic and, *in extenso*, information-theoretic attributes not describable, let alone explainable, by a classical theoresis of spacetime structure and its dynamics. It now seems natural to the theorist to anticipate that only a cogent quantum theory of gravity can deal effectively with black hole physics—especially with their aforementioned thermal evaporation phenomena and their horizons’ area-proportional entropy.

On the other hand, one could also view gravitational singularities from a slightly different perspective. Such a perspective was adopted by Finkelstein (Finkelstein, 1958) when he dealt with the doubly singular Schwarzschild solution to Einstein’s equations. In that paper he effectively showed, by employing a novel spacetime coordinate system, that the external singularity ($r = 2m$) in the “epidermis” of the Schwarzschild black hole indicates, in fact, that the latter is a unidirectional membrane allowing the propagation through its horizon of particles, but not of their antiparticles.¹²⁵ At the same time, however, his work also implicitly entailed that the interior singularity ($r = 0$) cannot be done away with simply by a coordinate transformation, thus indicating that in the “guts” of the Schwarzschild black hole—right at the point-mass source of the gravitational field—there is a “real” singularity (i.e. not just a coordinate one) which signals the inadequacy of general relativity in describing the gravitational field right at its source. Again, it is currently believed that only a quantum theoresis of gravity can achieve such a description.¹²⁶

To us, what is very educational from Finkelstein’s alternative perspective on the singularity riddle is the employment of new coordinates (albeit, still labelling the point events of a classical differential manifold model for spacetime) which effectively resolved the exterior singularity, followed by a sound physical interpretation (particle/antiparticle or past/future-asymmetry) of the resolved picture. This is in striking contrast to the usual treatment of singularities as real physical diseases that cannot be cured within the classical C^∞ -smooth differential geometric framework of general relativity (Penrose, 1977). Thus, in such “ C^∞ -conservative” approaches, singularities are not to be encountered, because one does not know how to treat them: rather, they are to be isolated and cut-off from a remaining “effective spacetime manifold” in which non-anomalous physical processes occur normally and can be adequately described by C^∞ -smooth means (Geroch 1968a,b,1972).

In contradistinction, what we advocate herein is akin, at least in spirit and philosophy, to Finkelstein’s approach: by changing focus from the classical coordinate

¹²⁵ Pointing thus to a fundamental time-asymmetry even in the classical gravitational deep (Finkelstein, 1988).

¹²⁶ This may be understood in close analogy with QED which effectively gave, with the aid of some theoretically rather *ad hoc* and conceptually questionable renormalization procedures, a calculationally finite theory of the interaction of the photon radiation field with its source point electron. Alas, quantum gravity, when regarded as the quantum field theory of $g_{\mu\nu}$, like QED is for A_μ , can be shown to be non-renormalizable . . .

structure algebra sheaf of C^∞ -smooth functions on the differential manifold to another structure \mathbb{A} -sheaf more suitable to the physical problem under theoretical scrutiny, while still retaining at our disposal most of the panoply of the powerful differential mechanism of the usual C^∞ -calculus, we effectively integrate, absorb or “engulf” singularities in our theory rather than stumble onto them and, as a result, meticulously try to avoid them.¹²⁷ Thus, altogether there is no issue of avoiding singularities or of continuing to perform C^∞ -calculus in a singularity-amputated smooth spacetime manifold, since we can calculate (i.e. actually carry out an abstract and quite universal calculus) in their very presence. Singularities are not impediments to ADG, for its abstract, algebraic in nature, sheaf-theoretic differential mechanism in a strong sense “sees through them,” while on the other hand, the classical C^∞ -differential geometry on smooth manifolds is quite impervious to and intolerant of them. What this contrast entails, of course, is that on the one hand mathematicians (especially differential equations specialists) should tell us what it means to set up a perfectly legitimate differential equation with possibly ultra-singular coefficient functions and look for its solutions within the ultra-singular structure \mathbb{A} -sheaf,¹²⁸ and on the other the theoretical physicist (especially the relativist and the quantum gravity researcher) is burdened with the responsibility to *physically interpret* ‘a dynamics amidst singularities, and in spite of them’ much in the same way that, as briefly noted above, Finkelstein (1958) physically interpreted the new picture of the exterior Schwarzschild singularity in the light of new coordinates as a semi-permeable (i.e., particle-allowing/antiparticle-excluding or equivalently past/future-asymmetric) membrane.¹²⁹

What will certainly burden us in the immediate future is to set up a finitary version of Einstein’s equations in the language of ADG, since they have already been cast abstractly (Mallios, in press; Mallios, in preparation). Indeed, the second author is already looking into this possibility (Raptis, in preparation); furthermore, it must be noted that the algebraic ideas propounded above are in close analogy with Regge’s famous coordinate-free and reticular simplicial gravity proposed Regge (1961) and further elaborated from a (topo)logical perspective in Zapatrin (1993),

¹²⁷ As it were, by making sure that we avoid them so that we can continue performing the usual C^∞ -calculus. In this sense our theory is not ‘ C^∞ -calculus conserving.’

¹²⁸ Like in Mallios and Rosinger (1999), for instance, where \mathbb{A} -sheaves of functions with everywhere dense singularities were studied under the prism of ADG, or even more generally, subsequently in Mallios and Rosinger (2001), where in the same spirit “multifoam algebras” dealing with singularities on arbitrary sets (under the proviso that their complements are dense) were considered.

¹²⁹ In a “psychological” sense, one is expected to be surprised or even intimidated, hence one’s calculus to be impeded, by singularities when one works in the featureless and uniform differential manifold and its C^∞ -algebras of coordinates; while on the other hand, if singularities is what one routinely encounters in the space and its coordinate functions that one is working with like, for instance, in Mallios and Rosinger (1999, 2001), and at the same time one is able to retain most of the practically useful differential mechanism, one is hardly in awe of singularities, so that one proceeds uninhibited with one’s differential geometric constructions and singularities present no essential problem.

although it must also be admitted that the “freedom from coordinates” espoused by ADG is, in fact, freedom to use in principle *any* coordinate algebra structure sheaf no matter how singular or anomalous it may seem to be from the conventional C^∞ -smooth perspective. For how can the laws of Nature, that are usually described in terms of differential equations, stumble upon out own measurements on our own coordinatizations (i.e. “arithmetizations” or “geometrizations”) of Her events (phenomena) and the spaces that host them? How can we ever hope to understand Physis if we ascribe to Her singularities and pathologies when it is more likely that it is our own theories that are short-sighted, of limited scope and descriptive power?

It seems only proper to us to conclude the present study as we stated it is Section 1, namely, by quoting and briefly commenting on Einstein, as well as by summarizing, by means of “sloganizing”, our basic thesis:

... It does not seem reasonable to introduce into a continuum theory points (or lines etc) of which the field equations do not hold... Is it conceivable that a field theory permits one to understand the atomistic and quantum structure of reality? Almost everybody will answer with “no” and ... at the present time nobody knows anything reliable about it ... so that we cannot judge in what manner and how strongly the exclusion of singularities reduces that manifold of solutions... *We do not possess any method at all to derive systematically solutions that are free of singularities.*¹³⁰ (1956). (Einstein, 1956)

We do sincerely hope that, at least conceptually, the ideas propounded herein will help us catch initial, but nevertheless clear, glimpses of such an apparently much needed mathematical method.

Finally, the following two “slogans” crystallize our central thesis in the present paper. In the same way that

Slogan 1. *Continuity is independent of the continuum.*¹³¹

so

Slogan 2. *Differentiability is independent of smoothness.*¹³²

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¹³⁰ Our emphasis.

¹³¹ That is, when spacetime is modelled after a topological (C^∞) manifold (Raptis, 2000b; Sorokin, 1991).

¹³² That is, when spacetime is modelled after a smooth (C^∞) manifold (Mallios, 1998a; Mallios and Raptis, 2001; Raptis and Zapatin, 2000, 2001).

with Steve Selesnick and Roman Zapatrin on the possibility of an entirely algebraic description of causality and how it may shed light on the problem of constructing a cogent quantum theory of spacetime structure and its dynamics (gravity). The incessant moral support of Jim Lambek over many years of research into sheaf and topos theory, and their potential application to theoretical physics, is also greatly appreciated. The second author is also grateful to the European Commission for the material support in the form of a generous Marie Curie Individual Research Fellowship held at Imperial College of Science, Technology and Medicine, London (UK).

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¹³³ There is also a Russian translation of this 2-volume book by MIR Publishers, Moscow (vol. 1, 2000 and vol. 2, 2001). See footnote 61 for relevant acknowledgments to the two Russian editors.

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¹³⁴ And extensive references about singularities and their theorems there.

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